Persistence and Path Dependence in the Spatial Economy*

Treb Allen
Dartmouth and NBER

Dave Donaldson
MIT and NBER

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Abstract

How much of the spatial distribution of economic activity today is determined by history rather than by geographic fundamentals? And if history matters for the distribution, does it also affect overall efficiency? This paper develops a tractable theoretical and empirical framework that aims to provide answers to these questions. We derive conditions on the strength of agglomeration externalities, valid for any geography, under which temporary historical shocks can have extremely persistent effects and even permanent consequences (path dependence). We also obtain new analytical expressions, functions of the particular geography in question, that bound the aggregate welfare level that can be sustained in any steady-state, thereby bounding the potential impact of history. Our simulations—based on parameters estimated from spatial variation across U.S. counties from 1800-2000—imply that small variations in historical conditions have substantial consequences for both the spatial distribution and the efficiency of U.S. economic activity, both today and in the long-run.

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1 Introduction

Economic activity in modern economies is staggeringly concentrated. For example, more than $1/6^{th}$ of value-added in the United States is currently produced in just three cities that occupy less than $1/160^{th}$ of its land area. But perhaps even more remarkable are the historical accidents that may have determined the location of these three cities—one was a Dutch fur trading post, one a *pueblo* for 22 adult and 22 children settlers designated by a Spanish governor to honor the angels, and one a river mouth known to Algonquin residents for its wild garlic (or, *chicago-ua*).

There is no shortage of examples in which the quirks of history appear to influence the current location of economic activity through either *persistence*—the long-lived dependence of current outcomes on temporary events—or *path dependence*—where temporary events fully govern long-run outcomes. See Nunn (2014) for a review. But how widespread should we expect these phenomena to be in the spatial economies around us? Going further, "does history matter only when it matters little?"—in Rauch's (1993) memorable phrase—because it merely reshuffles the current location of economic activity without much affecting aggregate efficiency?

In this paper we develop a new framework designed to shed light on these questions and then apply it to data from the United States between 1800 and 2000. We extend a rich vein of theoretical modeling (as synthesized in, for example, Fujita, Krugman, and Venables, 1999), in which agglomeration externalities can give rise to a potential multiplicity of equilibria, by adding overlapping generation dynamics and an arbitrary number of locations featuring general paths of geographic fundamentals and frictions. We derive conditions under which such an environment can feature substantial persistence and even path dependence. And our simulations, based on estimated parameter values, display exactly such phenomena for the U.S. spatial economy: even relatively minor historical shocks exhibit centuries-long dependence, and these shocks often lead to large and permanent differences in long-run aggregate welfare.

To arrive at this conclusion, we begin in Section 2 by stating four new theoretical results about this dynamic economic geography model. The first characterizes a condition for dynamic equilibria—that is, the transition paths that would take this economy from any starting point to any steady-state—to be unique, regardless of the underlying path of geographic fundamentals, as is important for the quantitative questions that we pose here. The second highlights how temporary shocks may be particularly persistent—that is, feature a very slow rate of convergence to a steady-state—when an economy gets close to the parameter threshold at which uniqueness is not guaranteed. Our third result characterizes necessary (and “globally” sufficient) conditions for the economy to feature multiple stable steady-states, which then creates the potential for path-dependent impacts of a temporary shock that could push an economy onto a permanently different path towards a distinct steady-state. Finally, our fourth result derives bounds on the aggregate welfare that is attainable across all possible steady-states, which is useful (given the numerical infeasibility of finding all steady-states in high-dimensional settings like ours) for distinguishing environments where path dependence has potentially large efficiency consequences.

The conditions in these four results all hinge on the strength of agglomeration forces (spillovers in production and amenities) relative to dispersion forces (agents' preferences for geographical diversity in trade and migration). Crucially, however, it is contemporaneous agglomeration spillovers that govern equilibrium uniqueness and the duration of persistence, whereas it is the sum of contemporaneous and historical spillovers that matters for the existence of multiple steady-states. This means that there exists a parameter range that features both well-behaved, unique transition paths as well as rich dynamic phenomena such as persistence and path dependence.

We therefore set out in Section 3 to estimate these parameters for the long-run spatial history of the United States from 1800-2000. Our estimating equations take the familiar form of a cross-location labor supply and demand system—as in the canonical Rosen-Roback tradition (Rosen, 1979; Roback, 1982; Glaeser, 2008) but augmented to allow for historical spillovers and interactions across locations due to costly trade and migration.

Despite this added empirical flexibility, parameter identification—even with an underlying potential for multiplicity—is still assured via familiar exclusion restrictions of the sort discussed by Roback (1982), albeit time-varying versions of these restrictions in our case. For the locational labor supply equation, which is identified from demand-side variation, we use shifters of agricultural productivity coming from the changing importance of certain crops over time and the advent of higher intensity cultivation methods. And for the locational labor demand equation we use shifters of the relevance of temperature extremes over time, which plausibly have changed the amenity value of certain locations, and hence labor supply, due to the development of technologies such as heating and air conditioning. Our estimates imply modest productivity spillovers, but an important role for positive historical spillovers on amenities—which are, as we show, consistent with models that feature durable locational investments, for example in housing.

Based on these parameter estimates we turn in Section 4 to a simulation exercise that is designed to shed light on the role that history plays in the modern-day U.S. spatial economy. Amidst the so-called “Technological Revolution” at the dawn of the 20th Century (c.f. Landes, 2003) it seems plausible that innovations such as electrification and the automobile had differential impacts across space for reasons that could be partially attributed to luck. For example, Henry Ford was born on a farm near Detroit, and Thomas Edison chose the 1901 Pan-American Exposition to demonstrate mass illumination via his new AC power, earning the host city of Buffalo its nickname, the "City of Light". Inspired by such anecdotes of happenstance, our counterfactual exercise asks what would have happened to the trajectories of two similar cities if their 1900 productivity fundamentals were randomly swapped, while holding all other conditions constant both before and after 1900. In practice, we pair locations on the basis of their 1900 population—for example, Buffalo (with a population of 436,000 in 1900) is paired with Cincinnati (412,000). In order to derive general lessons from such counterfactual swaps, we conduct one hundred simulations in which every location has an equal chance of either drawing its factual 1900 productivity or its counterfactual swap partner's 1900 productivity.

Even these relatively modest counterfactual swap histories turn out to have dramatic consequences. For example, across our simulations the median location has an elasticity of 0.89 between its population in 2000 and its population in 1900—so that a 10% drop in population due to an unfavorable but one-off productivity shock leaves the location about 9% smaller even a century later. And while trade and migration opportunities mean that the welfare of a location's residents is less affected by historical shocks, we find that the elasticity of realized welfare of adults residing in a location to historical shocks is still 0.21 for the median location. Simulating the economies forward into the future—undoubtedly a heroic exercise, but one that illustrates the workings of a model like ours—we find that the long arm of history reaches far into the future, with median population and welfare elasticities of 0.45 and 0.11, respectively, 500 years out.

Perhaps even more surprisingly, we find that these temporary historical shocks often have *permanent* effects on the spatial distribution and efficiency of the aggregate economy. That is, not only do our theoretical results imply that path dependence is possible, but our simulations find that even modest perturbations of history can knock the economy onto vastly different tracks. We find that the different counterfactual histories converge—though by no means uniformly—to (at least) three different steady-states, each with a distinct spatial distribution of economic activity. Moreover, the welfare gap between the steady-states is substantial: equivalent to a difference in growth rates of 0.25% per year for about 500 years. Simulating the evolution of the factual economy forward, we find that it ends up closer to the best counterfactual simulated steady-state than to the worst, although this still implies that long-run welfare in the factual economy could be more than 50% higher if the arrangement of spatial productivity within matched location pairs in 1900 had been slightly different. Finally, our analytical upper bound confirms that the steady-states probed by our swap-counterfactuals are close to the most efficient that any history could achieve, but the lower bound cannot rule out alternative historical conditions that may have been much much worse.

These findings shed new light on a number of strands of related work. First, we are inspired by an empirical literature that documents examples of spatial persistence and path dependence, or lack thereof, in the aftermath of historical events in a vast array of settings. Seminal work by Davis and Weinstein (2002, 2008) and Bleakley and Lin (2012, 2015) is emblematic of such lessons since Bleakley and Lin (2012, 2015) demonstrate long-lived (multi-century) persistence from long-obsolescent shipping technologies in the U.S. whereas Davis and Weinstein (2002, 2008) find that World War II bombing left only a relatively transitory (multi-decade) spatial trace in Japan. Wider examples from the U.S. alone include enduring impacts of slavery (Nunn, 2008), political boundaries (Dippel, 2014), flooding (Hornbeck and Naidu, 2014), mining activity (Glaeser, Kerr, and Kerr, 2015), fire damage (Hornbeck and Keniston, 2017), war destruction (Feigenbaum, Lee, and Mezzanotti, 2018), frontier exposure (Bazzi, Fiszbein, and Gebresilasse, 2020), and immigration (Sequeira, Nunn, and Qian, 2020)—among many other factors (see, e.g., Kim and Margo, 2014).[1]

Our findings clarify the conditions under which one could expect spatial persistence and path dependence to arise, which may rationalize the heterogenous effects seen in prior work. It can also provide a benchmark for the interpretation of studies that find persistent impacts of a given historical event and then aim to distinguish—a challenge for such work, as Nunn (2014) discusses—the hypothesis of a change to dynamics of fundamentals from the alternative that *any* temporary shock to fundamentals would have left a persistent geographic trace due to the logic of agglomeration and endogenous spatial lock-in.

Second, on the theory side, we draw on the insights of a literature that pioneered the understanding of path-dependent geographic settings. Krugman (1991), Matsuyama (1991), and Rauch (1993), for example, developed models with two locations and infinitely-lived agents. As fully elucidated in Herrendorf, Valentinyi, and Waldmann (2000) and Ottaviano (2001), the dynamics of equilibrium paths in such settings are dauntingly complex even in small-scale models, let alone in the high-dimensional empirical settings with realistic

[1]Further afield, just to cover a few examples, Dell (2010) documents persistent negative effects of forced labor institutions in Peru, Redding, Sturm, and Wolf (2011) uncover evidence for persistence in the location of airline hubs amidst the division and reunification of Germany, Jedwab and Moradi (2016) find persistent impacts of colonial railroads throughout most of Africa, Hanlon (2017) illustrates a long-lived spatial imprint resulting from the interruption of supplies to Britain's cotton textile industry cities during the U.S. Civil War, Henderson, Squires, Storeygard, and Weil (2018) describe how the differing extent to which physical geography attributes matter today for early and late developing countries is consistent with long persistence, Michaels and Rauch (2018) highlight the differing extents of persistence of Roman towns in England and France, and Dell and Olken (2020) document the enduring industrial development around sites of colonial investment in Indonesia.

geographies that are necessary for quantitative work. We have therefore endeavored to extract core lessons from these setups and adapt them to a framework that is amenable to the type of empirically-grounded quantification that is our goal. The new theoretical results that we derive regarding speed of convergence, uniqueness of dynamic paths, multiplicity of steady-states, and bounds on aggregate welfare across steady-states all work towards that objective. The main cost of this tractability is that we shut down multiplicity arising from forward-looking behavior, though with the relatively long time periods in our empirical analysis this restriction may not be especially limiting in practice.

Finally, we build on a recent body of work on quantitative economic geography models such as the static environments of Roback (1982), Glaeser (2008), Allen and Arkolakis (2014), Ahlfeldt, Redding, Sturm, and Wolf (2015)—summarized and synthesized in Redding and Rossi-Hansberg (2017)—as well as the pioneering dynamic models in Desmet, Nagy, and Rossi-Hansberg (2018), Caliendo, Dvorkin, and Parro (2019), and Nagy (2020). Our contribution is to extend these tools in order to facilitate the explicit study of geographic persistence and path dependence, to estimate, in the case of 200 years of U.S. economic geography, elasticities that our extended framework highlights as essential for such a theme, and then to apply the resulting estimates to counterfactual simulations about the consequentiality of historical shocks for the location and aggregate efficiency of economic activity in the U.S. today.

2 A dynamic economic geography framework

In this section we develop a dynamic economic geography framework that is amenable to the empirical study of geographic path dependence throughout U.S. history. A large set of regions possess arbitrary, time-varying fundamentals in terms of productivity and amenities. They interact via costly trade in goods and costly migration. Crucially, production and locational amenities both potentially involve contemporary and historical non-pecuniary spillovers—the forces behind both long persistence and path dependence.

2.1 Setup

There are $i \in \{1, ..., N\}$ locations and time is discrete and indexed by $t \in \{0, 1, ...\}$. Each individual lives for two periods. In the first period ("childhood"), a given individual is born where her parent lives and consumes what her parent consumes. At the beginning of the second period ("adulthood"), she realizes her own preferences and chooses where to live; then, in her chosen location, she supplies a unit of labor inelastically to produce, she consumes, and she gives birth to a child. Let L_{it} denote the number of workers (adults) residing in location i at time t, where the total number of workers $\sum_{i=1}^{N} L_{it} = \bar{L}$, is normalized to a constant in each period t.[2]

2.1.1 Production

Each location i is capable of producing a unique good—the Armington (1969) assumption. A continuum of firms (indexed by ω) in location i produce this homogeneous good under perfectly competitive conditions with the constant returns-to-scale production function $q_{it}(\omega) = A_{it} l_{it}(\omega)$, where labor $l_{it}(\omega)$ is the only production input, and hence $\int l_{it}(\omega) d\omega = L_{it}$. The productivity level for the location is given by

$$A_{it} = \bar{A}_{it} L_{it}^{\alpha_1} L_{it-1}^{\alpha_2}, \tag{1}$$

where $\bar{A}_{it}$ is the exogenous (but unrestricted) component of this location's productivity in year t. Importantly, the two additional components of a location's productivity in equation (1) depend on the number of workers in that location both in the current period, L_{it}, and in the previous period, L_{it-1}. We assume that firms take these aggregate labor quantities as given. Hence the parameter α_1 governs the strength of any potential (positive or negative) contemporaneous agglomeration externalities working through the size of local production. This is a simple way of capturing Marshallian externalities, external economies of scale, knowledge transfers, thick market effects in output or input markets, and the like. The presence of the term $L_{it}^{\alpha_1}$ is standard in many approaches to modeling spatial economies,[3] albeit typically in static models that would combine the effects of L_{it} and L_{it-1}.

The parameter α_2, on the other hand, governs the strength of potential *historical* agglomeration externalities. This allows for the possibility that two cities with equal fundamentals $\bar{A}_{it}$ and sizes L_{it} today might feature different productivity levels A_{it} today because they had differing sizes L_{it-1} in the past. There are many potential reasons that one might expect $\alpha_2 > 0$, and we describe two such sets of microfoundations briefly here (with complete derivations in Appendix B.1).

Consider first the potential persistence of local knowledge. In particular, we present a model based on Deneckere and Judd (1992), where firms can incur a fixed cost to develop a new variety, for which they earn monopolistic profits for a single period. In the subsequent period, the blueprint for the product becomes common knowledge so that the variety is produced under perfect competition, and we assume the product becomes obsolete (with

[2] Our model economy exhibits a form of scale-invariance that means that, for the purposes of our analysis here, the total number of workers in any time period is irrelevant for the distribution of economic activity.

[3] For example, see Redding and Rossi-Hansberg (2017).

zero demand for it) two periods after its creation. As in Krugman (1980), the equilibrium number of new varieties will be proportional to the contemporaneous local population. Given consumers' love of variety, new varieties act isomorphically to an increase in the productivity of the single Armington product, resulting in the precise form of equation (1) with $\alpha_1 \equiv \frac{\chi}{\rho-1}$ and $\alpha_2 \equiv \frac{1-\chi}{\rho-1}$, where χ is the expenditure share on all new varieties and $\rho > 1$ is the elasticity of substitution across individual varieties.

Second, consider the potential for durable investments in local productivity. In particular, we present a model based on Desmet and Rossi-Hansberg (2014), in which firms hire workers both to produce and to innovate, and where innovation increases each firm's own productivity contemporaneously and increases all firms' productivity levels in the subsequent period. If firms earn zero profits in equilibrium due to competitive bidding over a fixed factor (e.g. land), then, as in Desmet and Rossi-Hansberg (2014), the dynamic problem of the firm simplifies to a sequence of static profit-maximizing problems. With Cobb-Douglas production functions, equilibrium productivity can be written as equation (1) with $\alpha_1 \equiv \frac{\gamma_1}{\xi} - (1-\mu)$, and $\alpha_2 \equiv \delta\frac{\gamma_1}{\xi}$, where γ_1 governs the decreasing returns of innovation in productivity, ξ governs the decreasing returns of labor in innovation, δ is the depreciation of investment, and μ is the share of labor in the production function.

Of course, there are surely many sets of microfoundations that could generate the productivity spillover features assumed in equation (1). In what follows, we characterize the properties of the model and estimate the strength of the spillovers without taking a stand on the particular source of these externalities.

2.1.2 Consumption

An adult and her child consume with the same preferences, with a constant (but irrelevant) fraction allocated to the child. They have constant elasticity of substitution (CES) preferences, with elasticity $\sigma > 1$, across the differentiated goods that each location can produce. Letting w_{it} denote the equilibrium nominal wage, and letting P_{it} be the price index (solved for below), the deterministic component of welfare—that is, welfare up to an idiosyncratic shock that we introduce below—of any adult residing in location i at time t is given by

$$W_{it} \equiv u_{it}\frac{w_{it}}{P_{it}}, \tag{2}$$

where the component u_{it} refers to a location-specific amenity shifter that is given by

$$u_{it} = \bar{u}_{it}L_{it}^{\beta_1}L_{it-1}^{\beta_2}. \tag{3}$$

The term $\bar{u}_{it}$ allows for flexible exogenous amenity offerings in any location and time period. Endogenous amenities work analogously to the production externality terms introduced above, with the parameters β_1 and β_2 here capturing the potential for the presence of other adults in a location to directly affect (either positively or negatively, depending on the sign of β_1 and β_2) the utility of any given resident. We assume that consumers take these terms as given, just as they take factor and goods prices as given, when making decisions.

As is well understood, a natural source of a negative value for β_1 in a model such as this one is the possibility of local congestion forces that are not directly modeled here; for example, if non-tradable goods (such as housing and land) are in fixed supply locally and are demanded with fixed expenditure shares then $-\beta_1$ would equal the share of expenditure spent on such goods. Such effects would work contemporaneously, so they would govern β_1.

As with α_2, the parameter β_2 stands in for phenomena through which the historical population L_{it-1} affects the utility of residents in year t directly (that is, other than through productivity, wages, prices, or current population levels). Again it seems potentially important to allow for such effects given the likelihood that previous generations of residents may leave a durable impact, positive or negative, on their former locations of residence. Positive impacts could include the construction of infrastructure (e.g. parks, sewers, or housing), and negative impacts could include environmental damage or resource depletion.

It is straightforward to construct a model that generates exactly the specification of equation (3) for amenities. We sketch such a microfoundation here, and again present the complete set of derivations in Appendix B.2. We consider a model where agents consume both a tradable good and local housing, and each unit of land is owned by a real estate developer who bids for the rights to develop the land and then chooses the amount of housing to construct. To build housing, the developer combines local labor and the (depreciated) housing stock from the previous period. We assume the bidding process ensures developers earn zero profits, so as in Desmet and Rossi-Hansberg (2014) the dynamic problem of how much housing to construct simplifies into a series of static profit maximizing decisions. In equilibrium, the higher the contemporaneous population, the lower the utility of local residents (as the residents each consume less housing), whereas the higher the population in the previous period, the higher the utility of local residents (as more workers in the previous period results in a greater housing stock today). In particular, if production and utility functions are Cobb-Douglas (with μ the share of old housing in production and $1-\lambda$ the share of housing in expenditure) this model will be isomorphic to equation (3), with $\beta_1 = -\mu\frac{1-\lambda}{\lambda} < 0$ and $\beta_2 = \rho\mu\frac{1-\lambda}{\lambda} > 0$, where ρ is the depreciation rate of the housing stock.

As with the productivity spillovers, we emphasize that there may be other theoretical rationales for the amenity spillovers assumed in equation (3). In terms of what follows, there is no need to emphasize any one particular microfoundation.

2.1.3 Trade

Bilateral trade from location i to location j incurs an exogenous iceberg trade cost, $\tau_{ijt} \geq 1$ (where $\tau_{ijt} = 1$ corresponds to frictionless trade). Given this, bilateral trade flow expenditures X_{ijt} take on the well-known gravity form given by

$$X_{ijt} = \tau_{ijt}^{1-\sigma} \left(\frac{w_{it}}{A_{it}}\right)^{1-\sigma} P_{jt}^{\sigma-1} w_{jt} L_{jt}, \tag{4}$$

where

$$P_{it} \equiv \left(\sum_{k=1}^{N} \left(\tau_{ki} \frac{w_{kt}}{A_{kt}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{5}$$

is the CES price index referred to above.

For the empirical analysis below, it is convenient to write equation (4) as:

$$X_{ijt} = \tau_{ijt}^{-\theta} \times \frac{\left(Y_{it}/Y^W\right)}{\mathcal{P}_{it}^{1-\sigma}} \times \frac{Y_{jt}}{P_{jt}^{1-\sigma}}, \tag{6}$$

where

$$\mathcal{P}_{it} \equiv \left(\frac{w_{it}}{A_{it}}\right)^{-1} \left(\frac{Y_{it}}{Y^W}\right)^{\frac{1}{1-\sigma}}, \tag{7}$$

$Y_{it} \equiv w_{it} L_{it}$, and Y^W is total world income (which we normalize to one in what follows). In the terminology of the gravity trade literature (see e.g. Anderson and Van Wincoop, 2003), (the inverse of) $\mathcal{P}_{it}$ captures the *outward trade market access* of location i and (the inverse of) P_{jt} captures the *inward trade market access* of location j.

2.1.4 Migration

Recall from the discussion of timing above that L_{it-1} adults reside in location i at time $t-1$, and they have one child each. Those children choose at the beginning of period t—as they pass into adulthood—where they want to live as adults in order to maximize their welfare as adults.

As described above, adults who reside in a location j enjoy a deterministic component of utility given by W_{jt} in equilibrium, which we will refer to as *ex-post welfare*. Similarly to the trade costs introduced above, migrating from i to j costs $\mu_{ijt} \geq 1$ units of utility (so that frictionless migration is denoted by $\mu_{ijt} = 1$). This means that the deterministic utility enjoyed by a migrant who moves from location i to location j is $\frac{W_{jt}}{\mu_{ijt}}$. However, we also allow for idiosyncratic unobserved heterogeneity in how each child will value living in each location j in adulthood. Letting the vector of such idiosyncratic taste differences (one for each location) be denoted by $\vec{\varepsilon}$, the actual period payoff of a child who receives the draw $\vec{\varepsilon}$ while living in location i at time $t-1$ who chooses to move to location j as an adult is:

$$W_{ijt}\left(\vec{\varepsilon}\right) \equiv \frac{W_{jt}}{\mu_{ijt}}\varepsilon_j, \tag{8}$$

so the particular shock for location j, denoted by ε_j, simply scales up or down the deterministic component of utility, $\frac{W_{jt}}{\mu_{ijt}}$. Hence, any new adult chooses her location as follows:

$$\max_j W_{ijt}\left(\vec{\varepsilon}\right) = \max_j \frac{W_{jt}}{\mu_{ijt}}\varepsilon_j.$$

We further assume that $\vec{\varepsilon}$ is drawn independently from a Frechet distribution with shape parameter θ (and a set of location parameters that we normalize to one without loss). The number of children in location i at time $t-1$ who choose to move to location j at time t, L_{ijt}, is then given by:

$$L_{ijt} = \mu_{ijt}^{-\theta}\Pi_{it}^{-\theta}L_{it-1}W_{jt}^{\theta}, \tag{9}$$

where $\Pi_{it} \equiv \left(\sum_{k=1}^{N}\left(W_{kt}/\mu_{ikt}\right)^{\theta}\right)^{\frac{1}{\theta}}$ summarizes the appeal of the migration options from location i. Equation (9) says that there will be greater migration toward destination locations j with high ex-post welfare W_{jt} and low bilateral migration costs μ_{ijt}, and coming from origin locations i that either have a lot of residents L_{it-1} or poor outside options Π_{it}.

Finally, for the empirical analysis below, it is convenient to write equation (9) as:

$$L_{ijt} = \mu_{ijt}^{-\theta} \times \frac{L_{it-1}}{\Pi_{it}^{\theta}} \times \frac{L_{jt}/\bar{L}}{\Lambda_{jt}^{-\theta}}, \tag{10}$$

where

$$\Lambda_{it} \equiv W_{it}\left(\frac{L_{it}}{\bar{L}}\right)^{-\frac{1}{\theta}}. \tag{11}$$

As in the flow of goods above, Π_{it} captures the *outward migration market access* from i and the (inverse of) Λ_{jt} captures the *inward migration market access* to j. It turns out that both the outward and inward migration market access terms are closely related to the welfare of agents, a point we turn to next.

2.1.5 Welfare

Following a standard derivation, the expected adulthood period payoff of a child residing in location i at time $t-1$, prior to realizing her idiosyncratic shocks $\vec{\varepsilon}$, is equal to the outward migration access Π_{it}:

$$\mathbb{E}\left[\max_j W_{ijt}\left(\vec{\varepsilon}\right)\right] = \Pi_{it}. \tag{12}$$

Similarly, given an appropriate weighting of children whose parents came from different origins, we can express the period payoff of childhood for the average child residing in location i at period $t-1$ as equal to (the inverse of) the inward migration access:[4]

$$\left(\sum_j \left(\frac{L_{jit-1}}{L_{it-1}}\right) \mathbb{E}\left[W_{jit-1}\left(\vec{\varepsilon}\right)|i = \arg\max_j W_{jit-1}\left(\vec{\varepsilon}\right)\right]^{\theta}\right)^{\frac{1}{\theta}} = \Lambda_{it-1}. \tag{13}$$

Assuming agents aggregate payoffs in childhood and adulthood equally in a Cobb-Douglas fashion, the expected utility of an agent born in location i in period $t-1$, $\Omega_{i,t}$, is then simply the geometric mean of the outward and (inverse) inward migration accesses:

$$\Omega_{it} = \sqrt{\Lambda_{it-1} \times \Pi_{it}}. \tag{14}$$

As it combines both the realized payoffs of childhood and the expected payoffs of adulthood, in what follows we refer to Ω_{it} as *ex-ante welfare*.

2.2 Dynamic Equilibrium

An equilibrium in this dynamic economy is a sequence of values of (finite) prices and (strictly positive) allocations such that goods and factor markets clear in all periods.[5] More formally, for any strictly positive initial population vector $\{L_{i0}\}$ and geography vector $\{\bar{A}_{it}, \bar{u}_{it}, \tau_{ijt}, \mu_{ijt}\}$, an equilibrium is a vector of endogenous variables $\{L_{it}, w_{it}, W_{it}, \Pi_{it}\}$ such that, for all locations i and time periods t, we have:

[4] To account for the fact that children in i whose parents came from different origins inherit different expected idiosyncratic preferences, we consider a generalized weighted mean across parents of all possible origins, where the weights are their population shares in the destination and the generalized mean has the same power as the distribution of idiosyncratic preferences (so that the aggregation function for the welfare of agents across origins is the same as the aggregation function used by agents across destinations). We note that an alternative microfoundation, in which a "stork" allocates children to expectant parents based on the children's idiosyncratic preferences, would deliver an identical expression.

[5] Throughout, we confine attention to equilibria where all locations are inhabited, as (1) these are the empirically relevant types of equilibria at our geographic scale of analysis; and (2) in the presence of productivity and/or amenity spillovers, from equations (1) and (3), an uninhabited location will (trivially) remain uninhabited forever.

1. *Total sales are equal to payments to labor:* That is, a location's income is equal to the value of all locations' purchases from it, or $w_{it}L_{it} = \sum_j X_{ijt}$. Using equation (4) this can be written as

$$w_{it}^{\sigma} L_{it}^{1-\alpha(\sigma-1)} = \sum_j K_{ijt} L_{jt}^{\beta(\sigma-1)} W_{jt}^{1-\sigma} w_{jt}^{\sigma} L_{jt}, \tag{15}$$

with $K_{ijt} \equiv \left(\frac{\tau_{ijt}}{\bar{A}_{it} L_{it-1}^{\alpha_2} \bar{u}_{jt} L_{jt-1}^{\beta_2}} \right)^{1-\sigma}$ defined as a collection of terms that are either exogenous, or predetermined from the perspective of period t.

2. *Trade is balanced:* That is, a location's income is fully spent on goods from all locations, or $w_{it}L_{it} = \sum_j X_{jit}$. Using equation (4) this can be written as

$$w_{it}^{1-\sigma} L_{it}^{\beta_1(1-\sigma)} W_{it}^{\sigma-1} = \sum_j K_{jit} L_{jt}^{\alpha_1(\sigma-1)} w_{jt}^{1-\sigma}. \tag{16}$$

3. *A location's population is equal to the population arriving in that location:* That is, $L_{it} = \sum_j L_{jit}$. From equation (9) this implies

$$L_{it} W_{it}^{-\theta} = \sum_j \mu_{jit}^{-\theta} \Pi_{jt}^{-\theta} L_{jt-1}. \tag{17}$$

4. *A location's population in the previous period is equal to the number of people exiting that location:* That is, $L_{it-1} = \sum_j L_{ijt}$. From equation (9) this can be written as

$$L_{it-1} = \sum_j \mu_{ijt}^{-\theta} \Pi_{it}^{-\theta} L_{it-1} W_{jt}^{\theta},$$

which can then be written more compactly as

$$\Pi_{it}^{\theta} \equiv \sum_j \mu_{ijt}^{-\theta} W_{jt}^{\theta}. \tag{18}$$

Summarizing, the dynamic equilibrium can be represented as the system of $4 \times N \times T$ equations (in equations 15-18) in $4 \times N \times T$ unknowns, $\{L_{it}, w_{it}, W_{it}, \Pi_{it}\}$.[6]

This system of equations (15)-(18) comprises a high-dimensional nonlinear dynamic sys-

[6]When trade costs are symmetric (as will be assumed below) outward and inward goods market access $\mathcal{P}_{it}$ and P_{it} are equal up to scale, allowing equations (15) and (16) to be combined into a single non-linear equation, reducing the dimensionality of the system to $3 \times N \times T$ equations and $3 \times N \times T$ unknowns; see Anderson and Van Wincoop (2003) and Allen and Arkolakis (2014).

tem whose analysis can prove challenging. But this task is facilitated by the fact that the system is a collection of additive power equations, where each of the endogenous variables $\{L_{it}, w_{it}, W_{it}, \Pi_{it}\}$ appears, on either the left-hand or right-hand side, to a particular fixed power, with weights in the system given by an exogenous *kernel* term that comprises variables that are either exogenous or pre-determined from the perspective of period t (K_{ijt} in equations 15 and 16, and $\mu_{ijt}^{-\theta}$ in equations 17 and 18). This means that the solution of each cross-sectional system for t, given values of K_{ijt} and hence solutions from the previous period $t-1$, can be solved using the methods in Allen, Arkolakis, and Li (2020). In this manner, a dynamic path can be characterized by understanding a sequence of linked dynamic problems.

Towards this goal, we define the matrix

$$\mathbf{A}(\alpha, \beta) \equiv \begin{pmatrix} \left|\frac{\theta(1+\alpha\sigma+\beta(\sigma-1))-(\sigma-1)}{\sigma+\theta(1+(1-\sigma)\alpha-\beta\sigma)}\right| & \left|\frac{(\sigma-1)(\alpha+1)}{\sigma+\theta(1+(1-\sigma)\alpha-\beta\sigma)}\right| \\ \left|\frac{\theta/\tilde{\sigma}}{\sigma+\theta(1+(1-\sigma)\alpha-\beta\sigma)}\right| & \left|\frac{\theta(1-(\sigma-1)\alpha-\beta\sigma)}{\sigma+\theta(1+(1-\sigma)\alpha-\beta\sigma)}\right| \end{pmatrix}, \tag{19}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$. This notation stresses the dependence of $\mathbf{A}(\alpha, \beta)$ on α and β for reasons that will be made clear below. Given this definition, the following result characterizes a sufficient condition for existence and uniqueness for environments with symmetric trade costs (and unrestricted migration costs) and arbitrary positive geographic fundamentals.

Proposition 1. *For any initial population $\{L_{i0}\}$ and geography $\{\bar{A}_{it} > 0, \bar{u}_{it} > 0, \tau_{ijt} = \tau_{jit}, \mu_{ijt} > 0\}$, there exists an equilibrium. The equilibrium is unique if $\rho(\mathbf{A}(\alpha_1, \beta_1)) \leq 1$, where $\rho(\cdot)$ denotes the spectral radius (i.e. the largest eigenvalue in absolute value) operator.*

Proof. See Section A.1. □

This sufficient condition for uniqueness will be satisfied whenever α_1 and β_1 are sufficiently small. Panel (a) of Figure 1 illustrates this condition for the values of σ and θ that we use in our empirical calculations below. At these values, the sufficient condition of $\rho(\mathbf{A}(\alpha_1, \beta_1)) \leq 1$ is well approximated by the simple relation of $\alpha_1 + \beta_1 \leq 0$—that is, contemporaneous agglomeration forces must simply be non-positive on net. Finally, we note that this result concerning uniqueness of the dynamic equilibrium does not depend on the values of α_2 and β_2, since the current generation takes L_{it-1} as given.

To provide some intuition for the dynamic system, algebraic manipulations of equations (15)-(18) when trade costs are symmetric imply that the equilibrium distribution of population in any location and time can be written as

$$\gamma \ln L_{it} = C_t + \sigma \ln \bar{u}_{it} + (\sigma-1) \ln \bar{A}_{it} - (2\sigma-1) \ln P_{it} - \sigma \ln \Lambda_{it} + (\alpha_2(\sigma-1) + \beta_2\sigma) \ln L_{i,t-1}, \tag{20}$$

where $\gamma \equiv 1 + \frac{\sigma}{\theta} - (\alpha_1 (\sigma - 1) + \beta_1 \sigma)$ and C_t is a constant that ensures the aggregate labor market clearing condition holds.

Equation (20) has three implications: first, as long as $\gamma > 0$ (which corresponds to the case of our empirical estimates below), a greater density of residents can be found in any location with high productivity $\bar{A}_{it}$, high amenities $\bar{u}_{it}$, high inward migration access (low Λ_{it}), high access to imported goods (low P_{it}), and—if $\alpha_2 (\sigma - 1) + \beta_2 \sigma > 0$, so that historical spillovers are positive—with greater population density in the previous period. Second, the elasticities of the population to these characteristics are governed by the strength of γ, where greater contemporaneous spillover elasticities α_1 and β_1 result in larger population responses. Third, history—i.e. the distribution of the population in the previous period—only affects the current population through the inward market access terms (Λ_{it} and P_{it}) and through the direct impact on productivities and amenities from the historical spillover elasticities α_2 and β_2. Of course, while the first two determinants of population density in equation (20), $\bar{A}_{it}$ and $\bar{u}_{it}$, are exogenous in our model, the latter three determinants, Λ_{it}, P_{it}, and $L_{i,t-1}$ are endogenous and are determined simultaneously through interactions with the endogenous features in all other locations. It is the self-reinforcing potential of these interactions, both over time and across space, that leads to the rich dynamics that we explore below.

2.3 Persistence and Path Dependence

We now turn to a characterization of the dynamic properties of the model, namely the persistence of shocks to the economy and the possibility of multiple steady-states (i.e. the potential for path dependence).

Persistence

Consider first the question of persistence: how long does a temporary shock to the economy take to dissipate? To answer this question we define $\chi_{x,t} \equiv \frac{\max_i x_{i,t}/x_{i,t-1}}{\min_i x_{i,t}/x_{i,t-1}}$ to be the ratio of the maximum to minimum change in variable $x_{i,t}$ across all locations. Note that $\chi_{x,t} \geq 1$ and is equal to one if and only if $x_{i,t} \propto x_{i,t-1}$ for all i, i.e. the economy is on a balanced growth path (or, in our case where aggregate population is fixed, a steady-state). As such, it provides a convenient economy-wide measure of how far $x_{i,t}$ is from a steady-state. We can then define the economy-wide persistence of variable $x_{i,t}$ as the effect of $\chi_{x,t-1}$ on $\chi_{x,t}$—that is, how much deviations from the steady-state in period $t-1$ affect deviations from the steady-state in period t. The following proposition bounds the persistence of all endogenous outcomes in the model in this manner:

Proposition 2. *Consider any initial population* $\{L_{i0}\}$ *and time-invariant geography* $\{\bar{A}_i >$

$0, \bar{u}_i > 0, \tau_{ij} = \tau_{ji}, \mu_{ij} > 0\}$. *Suppose that* $\rho\left(\mathbf{A}\left(\alpha_{1,}, \beta_1\right)\right) < 1$ *so that from Proposition 1 the dynamic equilibrium is unique. Then the following relationship holds:*

$$\begin{pmatrix} \ln \chi_{L,t} \\ \ln \chi_{W,t} \\ \ln \chi_{\Pi,t} \end{pmatrix} \leq \left|\mathbf{B}^{-1}\right| \left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1, \beta_1\right)\right)^{-1} \mathbf{C} \left|\mathbf{B}\right| \begin{pmatrix} \ln \chi_{L,t-1} \\ \ln \chi_{W,t-1} \\ \ln \chi_{\Pi,t-1} \end{pmatrix}, \tag{21}$$

where $\tilde{\mathbf{A}}\left(\alpha, \beta\right) \equiv \begin{pmatrix} A_{11}\left(\alpha, \beta\right) & 0 & A_{12}\left(\alpha, \beta\right) \\ A_{21}\left(\alpha, \beta\right) & 0 & A_{22}\left(\alpha, \beta\right) \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{B} \equiv \begin{pmatrix} \tilde{\sigma}\left(1 - \alpha_1\left(\sigma - 1\right) - \beta_1\right) & \tilde{\sigma}\sigma & 0 \\ 1 & -\theta & 0 \\ 0 & 0 & \theta \end{pmatrix}$, $|\mathbf{B}|$ *indicates the element-wise absolute value of* $\mathbf{B}$, *and* $\mathbf{C}$ *is a 3-by-3 matrix whose first two rows are strictly positive (with values that depend on the parameters* $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma$ *and* θ*, as fully defined in Section A.2) and whose third row consists entirely of zeroes.*

Proof. See Section A.2. □

Proposition 2 provides an upper bound on how much the endogenous variables L_{it}, W_{it} and Π_{it} change form period $t-1$ to period t that depends on how much they changed from period $t-2$ to period $t-1$ (while holding constant the underlying geography in order to isolate the endogenous evolution of the economy). Loosely speaking, the proposition states that the closer the spillover parametersare to the boundary at which uniqueness can no longer be guaranteed, the greater the possibility of particularly long persistence. To see this, note that as the spectral radius $\rho\left(\mathbf{A}\left(\alpha_1, \beta_1\right)\right)$ approaches one from below, the largest eigenvalue of $\left(\mathbf{I} - \mathbf{A}\left(\alpha_1, \beta_1\right)\right)^{-1}$—and hence also the largest eigenvalue of $\left|\mathbf{B}^{-1}\right| \left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1, \beta_1\right)\right)^{-1} \mathbf{C} \left|\mathbf{B}\right|$—approaches infinity.[7] Panel (b) of Figure 1 illustrates this relationship by showing how the largest eigenvalue of $\left|\mathbf{B}^{-1}\right| \left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1, \beta_1\right)\right)^{-1} \mathbf{C} \left|\mathbf{B}\right|$ increases as α_1 and β_1 approach this boundary (holding constant σ, θ, α_2, and β_2 at the values we estimate in Section 3.3 below).

Path dependence

So far we have described the dynamic transition paths of this spatial economy. We now discuss the steady-state(s) to which these paths may converge. Intuitively, if local agglom-

[7] Noting that the eigenvalues of $\mathbf{A}\left(\alpha, \beta\right)$ are also eigenvalues of $\tilde{\mathbf{A}}\left(\alpha, \beta\right)$, this can be seen from a simple eigen-decomposition $\left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1, \beta_1\right)\right)^{-1} = \mathbf{V}'\Lambda\mathbf{V}$ where Λ is a diagonal matrix whose elements are the eigenvalues (including $\frac{1}{1-\rho\left(\mathbf{A}\left(\alpha_1, \beta_1\right)\right)}$, which approaches infinity as $\rho\left(\mathbf{A}\left(\alpha_1, \beta_1\right)\right)$ approaches one from below) and $\mathbf{V}$ is a 3×3 matrix of the associated eigenvectors. Note that because $\mathbf{A}\left(\alpha_1, \beta_1\right)$ is strictly positive and hence $\rho\left(\mathbf{A}\left(\alpha_1, \beta_1\right)\right) > 0$, the largest eigenvalue of $\left(\mathbf{I} - \mathbf{A}\left(\alpha_1, \beta_1\right)\right)^{-1}$ always exceeds unity, which indicates that long-lived persistence can never be ruled out.

eration economies are strong enough then there could be multiple allocations at which the economy would be in steady-state. Agents who come to reside in a location could find it optimal, on average, to stay there; and yet the same could simultaneously be true for another location, thanks to the reinforcing logic of local positive spillovers.

To evaluate this possibility we consider a version of the above economy but for which the potentially time-varying fundamentals $\left\{\bar{A}_{it}\right\}$ and $\{\bar{u}_{it}\}$ and trade $\{\tau_{ijt}\}$ and migration $\{\mu_{ijt}\}$ costs are constant over time at the values $\left\{\bar{A}_i, \bar{u}_i, \tau_{ij}, \mu_{ij}\right\}$. The steady-states of our economy will therefore be a set of time-invariant endogenous variables that we denote by $\{L_i, w_i, W_i, \Pi_i\}$.[8] The following result provides a sufficient condition for existence and uniqueness of the steady-state of this economy (for arbitrary geographies with symmetric trade and migration costs). It also shows how this is a maximal domain sufficient condition—the weakest condition one could impose whose result would be true for any geographic fundamentals.

Proposition 3. *For any time-invariant geography* $\left\{\bar{A}_i > 0, \bar{u}_i > 0, \tau_{ij} = \tau_{ji}, \mu_{ij} = \mu_{ji}\right\}$*, there exists a steady-state equilibrium and that equilibrium is unique if* $\rho\left(\mathbf{A}\left(\alpha_1 + \alpha_2, \beta_1 + \beta_2\right)\right) \leq 1$*. Moreover, if* $\rho\left(\mathbf{A}\left(\alpha_1 + \alpha_2, \beta_1 + \beta_2\right)\right) > 1$*, then there exist many geographies for which there are multiple steady-states at each geography.*

Proof. See Section A.3. □

The condition for uniqueness of the steady-state in Proposition 3 is similar to that for uniqueness of transition paths in Proposition 1. The only difference is that the latter condition depends on the size of contemporaneous spillovers α_1 and β_1, whereas the former condition depends on the size of total (that is, contemporaneous plus historical) spillovers $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$. This highlights the importance of the $\mathbf{A}\left(\alpha, \beta\right)$ matrix defined in equation (19). The second part of Proposition 3 demonstrates that the sufficient condition for uniqueness is necessary for certain geographies. Indeed, the proof of this proposition provides a continuum of example geographies under which multiple steady-states arise.

Associated with each steady-state is a *basin of attraction*: a set of values of the initial population distribution $\{L_{i0}\}$ for which the economy will converge to the steady-state in question. When there are multiple steady-states, and hence multiple basins of attraction, the eventual steady-state equilibrium of the economy will generically depend on its initial population distribution. Such a situation offers the potential for *path dependence*: where historical events that determine $\{L_{i0}\}$ can have permanent effects on the economy's outcomes since they select the basin of attraction in which populations are distributed at time 0, and

[8] Note that while population levels at each location L_i are constant in steady-state, and hence net migration flows are zero, gross migration flows are still positive in a steady-state equilibrium due to the churn induced by the idiosyncratic locational preferences in equation (8).

hence the eventual steady-state that is reached. Since the dynamic equilibria described in equations (15)-(18) feature a historical dependence on the state variable $\{L_{it}\}$ with only one lag, this means that from the perspective of any date t the "history" of the system (all exogenous and endogenous outcomes in the past) is fully characterized by $\{L_{i,t-1}\}$. Hence, observing the phenomenon that some event had a path-dependent impact hinges on whether the event moved $\{L_{i,t-1}\}$ across the boundary from one basin of attraction to another. We explore this feature in our counterfactual simulations in Section 4.

Combining Propositions 1 and 3, we see that the historical spillover parameters α_2 and β_2 play an important role in the study of path-dependent economies. Proposition 1 states that when the contemporaneous spillover parameters α_1 and β_1 are low then dynamic equilibrium paths will be unique. However, Proposition 3 states that when $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$ are high then steady-states are likely to be multiple. In this range of parameters (that is, with relatively low α_1 and β_1 and yet relatively high α_2 and β_2) path dependence can occur and yet be straightforward to study since the complications (for estimation, computation, and interpretation of counterfactuals) of genuine equilibrium indeterminacy do not arise.

Steady-state welfare bounds

The possibility of multiple steady-states highlighted by Proposition 3 raises the question of whether one steady-state is superior, in welfare terms, to others that could be reached from different initial conditions. To describe this possibility requires defining a particular notion of aggregate welfare. In the steady-state, it turns out that ex-ante welfare Ω_i is equalized across all locations:

$$\Omega_i = \Omega \ \forall i \in \{1, ..., N\},$$

so that Ω becomes a natural measure of the aggregate efficiency of a particular steady-state equilibrium.

Our penultimate proposition provides bounds on the aggregate welfare level Ω across all steady-states that can arise for a given geography (with symmetric trade and migration costs). Such bounds serve two purposes. First, they constrain the possible welfare impacts of history in the long-run without having to explicitly calculate all possible steady-state equilibria—a process that is infeasible in real-world settings with many locations and complex geographies. Second, the bounds provide an insight into how features of the underlying geography may exacerbate or attenuate the welfare impacts of history.

These bounds apply when the sum of all spillovers, $\rho \equiv \alpha_1 + \alpha_2 + \beta_1 + \beta_2$, is sufficiently strong to possibly generate multiple steady-states, but not so strong as to result in complete concentration of economic activity in one location. Under these conditions (which require the stated parameter restrictions), the following proposition provides a relationship between the geography of the economy and the possible values that the steady-state welfare Ω can take.

Proposition 4. *Consider any time-invariant geography* $\left\{\bar{A}_i > 0, \bar{u}_i > 0, \tau_{ij} = \tau_{ji}, \mu_{ij} = \mu_{ji}\right\}$ *and suppose that* $\rho > \max\left(0, \frac{1}{\theta} - \frac{1}{\sigma-1}\right)$ *and* $\frac{(1+(\alpha_1+\alpha_2)\sigma+(\beta_1+\beta_2)(\sigma-1))}{(1-(\alpha_1+\alpha_2)(\sigma-1)-\sigma(\beta_1+\beta_2))} < 1$*. Then the equilibrium welfare values* Ω *across all steady-states are bounded by*

$$\underline{\Omega} \leq \Omega \leq \bar{\Omega},$$

where the upper bound is given by

$$\bar{\Omega} \equiv \sqrt{c_1 \times \bar{\lambda}_M^{\frac{1}{\theta}} \times \bar{\lambda}_T^{\frac{1}{\sigma-1}} \times \max_i \bar{A}_i \bar{u}_i \times \bar{L}^{\rho-\frac{1}{\theta}} \times N^{\varepsilon_1}}, \tag{22}$$

the lower bound is given by

$$\underline{\Omega} \equiv \sqrt{c_2 \times \underline{\lambda}_M^{\frac{1}{\theta}} \times \underline{\lambda}_T^{\frac{1}{\sigma-1}} \times \min_i \bar{A}_i \bar{u}_i \times \bar{L}^{\rho} \times N^{-\varepsilon_2}}, \tag{23}$$

$\bar{\lambda}_M$ *and* $\underline{\lambda}_M$ *are the maximal and minimal eigenvalues (in absolute value) of the migration matrix* $\mathbf{M} \equiv \left[\mu_{ij}^{-\theta}\right]$, $\bar{\lambda}_T$ *and* $\underline{\lambda}_T$ *are the maximal and minimal eigenvalues (in absolute value) of the trade matrix* $\mathbf{T} \equiv \left[\tau_{ij}^{1-\sigma}\right]$, $\varepsilon_1 \equiv \frac{1}{2}\left(\frac{1}{\sigma-1} + \frac{1}{\theta}\right) + \mathbf{1}\left\{\rho < \frac{1}{2(\sigma-1)}\right\} \frac{1-2(\sigma-1)\rho}{2(\sigma-1)} > 0$, $\varepsilon_2 \equiv \left(\rho + \frac{3}{2(\sigma-1)} + \frac{1}{2\theta}\right) > 0$, *and* c_1 *and* c_2 *are constants (defined in Section A.4) that bound the variation in ex-post welfare* W_i *across locations (such that if* $\theta \to \infty$ *and hence* W_i *is equalized across all locations, then* $c_1 = c_2 = 1$*).*

Proof. See Section A.5. □

As the counterfactual simulations in Section 4 illustrate, when the presence of agglomeration forces results in multiple steady-states, different initial conditions can lead to different steady-states with different associated levels of aggregate welfare. Proposition 4 elucidates the scope for such differences by providing upper and lower bounds to all levels of steady-state welfare that are possible for a given geography.

The bounds in Proposition 4 also provide an intuitive explanation for how each component of geography can matter for welfare. The upper bound is the product of six terms: (i) the largest eigenvalue of the migration costs matrix (scaled by the migration elasticity), which generally rises as migration costs fall; (ii) an analogous term for the trade costs matrix (scaled by the trade elasticity); (iii) the innate productivity and amenity of the best location; (iv) the total labor endowment, scaled by the strength of net agglomerative forces ($\rho > 0$) and tempered by diminished utility from people concentrating in locations for which they may not have an idiosyncratic preference; (v) the total number of locations (scaled by locational differentiation via trade and migration); and (vi) a term capturing the variation across locations in W_i, which is included for technical reasons that—loosely speaking—arise because, ceteris paribus, individuals migrate to destinations with higher ex-post welfare but purchase goods from destinations with lower ex-post welfare.

The lower bound includes similar terms to the upper bound but with the logic inverted: the eigenvalues are now the smallest (in absolute value) of the trade and migration matrix, the measure of the overall quality of the innate productivities and amenities places are those of the worst location, and the number of locations is now scaled negatively by the strength of the agglomerative forces.

Proposition 4 characterizes how geography can shape the extent to which history matters for welfare in the long-run. To see this, define $\hat{\Omega}^{PD}$ as the ratio of the best to the worst possible steady-state welfare, holding geography fixed. Proposition 4 immediately implies

$$\hat{\Omega}^{PD} \leq c\sqrt{\kappa\left(\mathbf{M}\right)^{\frac{1}{\theta}} \times \kappa\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \times \frac{\max_i \bar{A}_i \bar{u}_i}{\min_i \bar{A}_i \bar{u}_i} \times \bar{L}^{-\frac{1}{\theta}} \times N^{\varepsilon_1+\varepsilon_2}}, \tag{24}$$

where $\kappa\left(\mathbf{A}\right)$ is the condition number of matrix $\mathbf{A}$. Recall that the condition number of a matrix measures how sensitive the solution to the linear equation $\mathbf{Ax} = \mathbf{b}$ is to approximation (where $\kappa\left(\mathbf{A}\right) = 1$ only if $\mathbf{A}$ is a scalar multiple of a linear isometry and $\kappa\left(\mathbf{A}\right) = \infty$ only if $\mathbf{A}$ is singular). Loosely speaking, equation (24) says that the welfare cost of history is bounded above by the sensitivity of the matrices of migration and trade costs.

Finally, we note that the upper and lower bounds provided here may not necessarily be tight for a given geography. This is clear from the nature of equation (24), which serves to decompose the sources of welfare variation across multiple steady-states into separate terms for each component of geography (i.e. trade costs, migration costs, local productivities and amenities, aggregate labor endowments, and the number of locations). Steady-state welfare levels are driven by the combination of each of these forces, and so no attempt to divide them up into separate contributions, as in equation (24), could ever provide tight bounds in general.[9] Whether the bounds are quantitatively useful will of course depend on the context; fortunately, we will see below that they are informative in our empirical setting.

[9] As an example, consider the case where migration is costless, i.e. $\mu_{ij} = 1$ for all i and j. In this case, the steady-state of the model corresponds to the equilibrium of a large class of (static) spatial models, including, for example, Helpman (1998), Allen and Arkolakis (2014), and Redding (2016). In this case, however, the condition number $\kappa\left(\mathbf{M}\right)^{\frac{1}{\theta}}$ is infinite, as the migration cost matrix is singular. However, it can be shown using a similar (but simpler) argument following the proof of Proposition 4 that in this case, the bounds of Proposition 4 still hold by excluding $\bar{\lambda}_M$ and $\underline{\lambda}_M$ from the bound expressions.

2.4 An example

To see the implications of Propositions 2, 3, and 4 more concretely, consider a simple economy of three locations. Suppose, to begin, that these locations have identical and time-invariant fundamentals $\{\bar{A}_{it}, \bar{u}_{it}, \tau_{ijt}, \mu_{ijt}\}$, and trade and migration costs are symmetric across locations.[10] Figure 2 shows phase diagrams on the two-dimensional space of L_{it} shares in this economy. The blue rays indicate one period of movement (so a ray's length shows speed of adjustment) in the direction towards each red dot and yellow stars denote steady-states.

We begin by illustrating how historical persistence is shaped by model parameters in panel (a). The left diagram shows the phase diagram with strong congestion forces ($\alpha_1 = -0.25$) so that the economy is far from the boundary of non-uniqueness. The right diagram, in contrast, shows an economy with weak congestion forces ($\alpha_1 = 0$), moving the economy closer to the boundary of non-uniqueness. As evinced by the shorter arrows, moving closer to the boundary of non-unique dynamics increases persistence—all dynamics of adjustment toward the unique steady-state will be slower—consistent with Proposition 2.

Panel (b) demonstrates how increasing the historical agglomeration forces creates the possibility of path dependence (consistent with Proposition 3). In the left diagram (which is identical to the right diagram of panel (a)), there are no historical agglomeration spillovers ($\alpha_2 = 0$), and the economy converges everywhere to the unique steady-state with equal population in all locations. In the right diagram, we increase the strength of historical agglomeration forces ($\alpha_2 = 0.25$), resulting in a dramatic change in the qualitative dynamic patterns. While the symmetric allocation remains a steady-state, it is no longer stable, and now there are three stable steady-states with relatively concentrated population shares near the corners of the simplex.[11] Also apparent are the three basins of attraction around these stable steady-states.

We turn next to the welfare consequences of path dependence, using panel (c) to illustrate. In the left diagram (which is identical to the right diagram of panel (b)), all locations have identical innate productivities and amenities; as a result, each of the three stable steady-states delivers the same level of aggregate welfare Ω. The right diagram introduces asymmetry in *local* geography by endowing location 1 with a slightly higher innate amenity ($\bar{u}_1 = 1.01$, $\bar{u}_2 = \bar{u}_3 = 1$). Consistent with Proposition 3, there remain multiple steady-states but now aggregate welfare is higher in the steady-state that achieves concentration in location 1. This is consistent with Proposition 4, which states that the upper bound on the welfare differences

[10] Throughout all economies in Figure 2, we set $\sigma = 9$, $\theta = 4$, $\beta_1 = \beta_2 = 0$, $\bar{A}_i = 1$ for all i, and $\mu_{ij} = 1.6$ for all $i \neq j$ and $\mu_{ii} = 1$.

[11] In addition to the unstable steady-state in the center, there are three additional unstable steady-states with equal concentration in two of the three locations (and almost no population in the third).

across steady-states is increasing in the ratio of the product of innate productivities and amenities across the best and worst locations. In this example the economy could start with a bad set of initial conditions and end up in a dominated steady-state. Reassuringly, however, the basin of attraction of the relatively good steady-state is larger than that of the dominated ones. So, in the space of all possible initial conditions, good steady-states are more likely to arise.

Finally, in panel (d) of Figure 2 we illustrate how the welfare consequences of path dependence change with *global* geography (i.e. economy-wide trade and migration costs). In the left diagram (which is identical to the right diagram of panel (c)), trade costs between locations are high ($\tau = 1.1$). In the right panel, we lower bilateral trade costs ($\tau = 1.08$). While reducing trade costs increases the aggregate welfare of the economy in all steady-states, it does so disproportionately more for the "good" steady-state where all individuals concentrate in location 1 than the steady-states with concentration in location 2 or 3. This means that the welfare consequences of path dependence have increased. Since falling trade costs are associated with an increase in the condition number of the trade cost matrix $\kappa(\mathbf{T})$, this is consistent with the predictions of Proposition 4.

3 Identification and Estimation

We now describe a procedure for mapping the above model into observable features of the U.S. economy throughout the past two centuries. The goal is to estimate the elasticity parameters (α_1, β_1, α_2, β_2, σ and θ) and geographic fundamentals $\left\{\bar{A}_{it}, \bar{u}_{it}, \tau_{ijt}, \mu_{ijt}\right\}$ that are critical for assessing the strength of persistence and likelihood of path dependence.

3.1 Data

Our quantification requires data on population L_{it} and per-capita nominal incomes w_{it}. We therefore build a dataset drawing on Manson, Schroeder, Van Riper, Ruggles, et al. (2017) that tracks these two variables for subnational regions i of the coterminous U.S. for as long a history as possible.

Starting with L_{it}, we obtain this series from decennial Census records of county-level population (by age group) from 1800 onward. To distinguish between children and adults in the model, we consider persons aged 25-74 as adults and work with 50-year steps (1800, 1850, 1900, 1950 and 2000) in order to avoid overlaps of these cohorts. Turning to w_{it}, for the years 1850-1950 we proxy for the relative amount of total income in any location, $w_{it}L_{it}$, by the estimated value of county-level agricultural and manufacturing output; for 2000 we use the per-capita income reported in the Census.[12] As a result, we have proxies for L_{it} and w_{it} from 1850-2000 and for L_{it} in 1800 as well; this allows estimation to proceed from 1850 onwards.[13]

To account for county border changes over the years we work with the set of (the largest possible) sub-county regions, each denoted by i, that can be mapped uniquely to every county in our five years of data.[14] In the end, our sample consists of 4,975 such sub-county regions i, which we refer to as "locations" from now on.

Three other variables—proxies for migration flows, trade flows and productivities and amenities—play an auxiliary role in our model estimation and are described further below.

3.2 Identification and Estimation

We now describe a three-step procedure that estimates the unknown parameters of the model in Section 2. In a nutshell, the third step involves estimating a system of locational labor supply and demand equations that represent an augmented version of the spatial equilibrium model due to Rosen and Roback (Rosen, 1979; Roback, 1982); the first and second steps simply prepare the ingredients necessary to proceed in this standard tradition.

The goal of the *first step* is to determine the level of the trade and migration cost terms, raised to their respective elasticity exponents, that enter the equilibrium system of equations (15)-(18), objects that we define as $T_{ijt} \equiv \tau_{ijt}^{1-\sigma}$ and $M_{ijt} \equiv \mu_{ijt}^{-\theta}$.[15] We do this using the available data on intranational trade X_{ijt} and migration L_{ijt} in our context: the 1997 Commodity Flow Survey (CFS), which measures trade flows; and Census data from 1850 onwards documenting the state of birth of each respondent (aged 25-74), which corresponds to the timing of migration in our model.[16] We then project these trade and migration cost terms onto bilateral distance (denoted $dist_{ij}$), in logs, as $\ln T_{ijt} = \kappa \ln dist_{ij}$, for all t, and

[12] In practice, manufacturing output is not available for 1950 so we use the 1940 value of agricultural and manufacturing output. Per-capita income is not readily available prior to 1980.

[13] Appendix Figures C.1 and C.2 present maps of L_{it} and w_{it} in all years.

[14] For example, suppose that county "A" in 1900 splits into "A1" and "A2" by 1950, and then "A2" splits into "A2(i)" and "A2(ii)" by 2000. The resulting sub-county regions that we track throughout would be "A1", "A2(i)" and "A2(ii)". We then apportion the county-level information into each of the sub-country regions on the basis of land area shares (and cluster regression standard errors at the county-year level).

[15] We note that only bilateral-specific elements of such terms matter in this system because origin- or destination-specific components would be redundant conditional on the unrestricted components A_{it} and u_{it}. We therefore normalize any origin-time and destination-time components of T_{ijt} and M_{ijt} to one.

[16] We aggregate both trade and migration flows to the state-to-state level, as this is the greatest level of disaggregation possible that is consistent between the two datasets. This aggregation introduces the measurement errors ε_{ijt} and ν_{ijt} in equations (25) and (26), which we assume are uncorrelated with distance. We note that Monte, Redding, and Rossi-Hansberg (2018) find the aggregation bias from applying gravity regressions on the CFS data (at the CFS area level) to county level data to be small.

$\ln M_{ijt} = \lambda_t \ln dist_{ij}$.[17] Substituting these expressions into the gravity equations for trade and migration flows, equations (6) and (10) respectively, we obtain

$$\ln X_{ijt} = \kappa \ln dist_{ij} + \gamma_{it} + \delta_{jt} + \varepsilon_{ijt}, \tag{25}$$

$$\ln L_{ijt} = \lambda_t \ln dist_{ij} + \rho_{it} + \pi_{jt} + \nu_{ijt}, \tag{26}$$

where the terms γ_{it}, δ_{jt}, ρ_{it}, and π_{jt} represent fixed effects.

Turning to our *second step,* we treat $\widehat{T}_{ijt} = (dist_{ij})^{\widehat{\kappa}}$ and $\widehat{M}_{ijt} = (dist_{ij})^{\widehat{\lambda}_t}$ as known from step one. Then, re-writing the equilibrium system of equations (15)-(18) using equations (6) and (10) yields, for all i:

$$\mathcal{P}_{it}^{1-\sigma} = \sum_j \widehat{T}_{ijt} \times Y_{jt} \times \left(P_{jt}^{1-\sigma}\right)^{-1}, \tag{27}$$

$$P_{it}^{1-\sigma} = \sum_j \widehat{T}_{jit} \times Y_{jt} \times \left(\mathcal{P}_{jt}^{1-\sigma}\right)^{-1}, \tag{28}$$

$$\left(\Lambda_{it}^{\theta}\right)^{-1} = \sum_j \widehat{M}_{jit} \times L_{jt-1} \times \left(\Pi_{jt}^{\theta}\right)^{-1}, \tag{29}$$

$$\Pi_{it}^{\theta} = \sum_j \widehat{M}_{ijt} \times L_{jt} \times \Lambda_{jt}^{\theta}. \tag{30}$$

The following proposition shows that the four remaining unknown variables—comprising the inward and outward trade and migration market access terms—in equations (15)-(18) are identified (to scale), when raised to the exponents $\sigma - 1$ or θ, because this system of equations has a unique solution given data on Y_{it}, L_{it} and estimates of $\widehat{T}_{ijt}$ and $\widehat{M}_{ijt}$ from step one.

Proposition 5. *Given observed data on* $\{Y_{it}, L_{it}, L_{it-1}\}$ *and given values of* $\left\{\widehat{T}_{ijt}, \widehat{M}_{ijt}\right\}$ *there exists a unique (up to scale) set of values of* $\left\{\mathcal{P}_{it}^{\sigma-1}, P_{it}^{\sigma-1}, \Pi_{it}^{\theta}, \Lambda_{it}^{\theta}\right\}$ *that satisfy equations (27)-(30).*

Proof. See Section A.5. □

Finally, we turn to the *third step* of our estimation procedure. Substituting the productivity spillover function from equation (1) into the outward trade market access $\mathcal{P}_{it}$ from equation (7) and imposing $Y_{it} = w_{it} L_{it}$, we obtain the following (inverse) demand equation

[17] This assumes (due to the fact that the CFS is only available in recent years) that κ is constant over time, as is broadly consistent with the patterns in international trade data surveyed by Disdier and Head (2008).

for labor in location i:

$$\ln w_{it} = \left[\alpha_1\left(\frac{\sigma-1}{\sigma}\right) - \frac{1}{\sigma}\right]\ln L_{it} + \alpha_2\left(\frac{\sigma-1}{\sigma}\right)\ln L_{it-1} + \frac{1}{1-\sigma}\ln \mathcal{P}_{it}^{1-\sigma} + \frac{\sigma-1}{\sigma}\ln \overline{A}_{it}. \tag{31}$$

In this expression, the inverse elasticity of labor demand combines the inverse elasticity of demand for goods from a location, $-\frac{1}{\sigma}$, with the contemporaneous productivity spillovers α_1; this latter effect is moderated by $\frac{\sigma-1}{\sigma}$ because the location faces a downward-sloping demand curve for its output. Notably, with strong positive spillovers the labor demand curve can be upward-sloping. Also present in the demand equation are a set of shifters: (i) lagged population L_{it-1}, which raises productivity if there are historical productivity externalities (i.e. $\alpha_2 > 0$); (ii) the outward trade market access $\mathcal{P}_{it}$, which allows for the labor demand in location i to be high if its ability to sell goods to other locations is high; and (iii) the exogenous (and unobserved) component of productivity, $\overline{A}_{it}$. An important feature of this estimating equation is that it describes a cross-sectional relationship that holds within any equilibrium, so it can be used for valid point estimation even if the model's parameters lie in a region for which multiplicity is possible.

Similarly, substituting the amenity spillover function from equation (3) into the inward migration market access Λ_{it} from equation (11) and using $W_{it} \equiv \frac{w_{it}}{P_{it}}u_{it}$, we obtain the following (inverse) supply equation for labor in location i:

$$\ln w_{it} = \left(\frac{1}{\theta} - \beta_1\right)\ln L_{it} + (-\beta_2)\ln L_{it-1} + \frac{1}{\theta}\ln \Lambda_{it}^{\theta} + \frac{1}{1-\sigma}\ln P_{it}^{1-\sigma} - \ln \overline{u}_{it}. \tag{32}$$

The inverse elasticity of labor supply combines the locational utility heterogeneity dispersion θ with the contemporaneous productivity spillovers β_1; analogously to the demand case, the elasticity of labor supply can be negative if such spillovers are positive and large. Shifters of the inverse labor supply curve comprise: (i) lagged population in the location L_{it-1}, which matters to the extent that historical amenity externalities (i.e. $\beta_2 \neq 0$) exist; (ii) the consumer cost-of-living P_{it}, which increases the nominal wage w_{it} that is required for a given amount of mobile workers to be willing to live in location i; (iii) the inbound supply of potential migrants from other nearby locations as captured by Λ_{it}; and (iv) the exogenous (and unobserved) component of location i's amenity, $\overline{u}_{it}$. Again, this equation allows parameter estimation to proceed even though equilibria may be multiple.

The locational demand-supply system in equations (31) and (32) generalizes that in the Rosen-Roback framework (c.f. Glaeser and Gottlieb 2009; Kline and Moretti 2014; Hsieh and Moretti 2019) in two senses. First, it relaxes the assumption that locations produce a homogenous and freely traded product (i.e. that σ is infinite and $\tau_{ijt} = 1$). Second, it relaxes the assumption that all workers have identical preferences across locations and face no costs of migrating (i.e. that θ is infinite, and $\mu_{ijt} = 1$). This added flexibility necessitates the inclusion of the market access terms $\mathcal{P}_{it}^{1-\sigma}$, $P_{it}^{1-\sigma}$ and Λ_{it}^{θ} as demand and supply shifters, as recovered in step two.

As with any demand-supply system, OLS estimates of the parameters in equations (31) and (32) would typically suffer from simultaneity bias. We therefore use an instrumental variable (IV) procedure that exploits the model's feature that exogenous shifters of amenities (components of $\overline{u}_{it}$) would be valid instruments for estimating the demand equation (31) and exogenous shifters of productivities (components of $\overline{A}_{it}$) would be valid instruments for estimating the supply equation (32) as long as those shifters are orthogonal to each other. In practice, we include location and region-year fixed effects in these IV specifications so that valid instruments for the case of the demand equation (31) are those that capture changes in amenities over time (within region) but do not relate to changes in the exogenous component of productivity.[18] Analogously, the supply equation (32) requires instruments derived from changes in productivity that are uncorrelated with changes in amenities. Given the 50-year time intervals that we use for estimation, these instruments must derive from relatively long-run changes to the U.S. economy.

For the demand equation, we follow Barreca, Clay, Deschenes, Greenstone, and Shapiro (2016) who note that technological advances like air conditioning and more effective heating systems have made extreme hot or cold climates more bearable (delivering greater amenity value) throughout our sample period. Accordingly, our IVs consist of a linear time trend interacted with the average maximum temperature in the warmest month and the average minimum temperature in the coldest month (and their squared values to allow for nonlinearities) in each location. We obtain such data from WorldClim.org.

For the labor supply equation instruments, we leverage two major changes in U.S. agriculture over the past 200 years. The first is the increased use of more intensive cultivation practices (e.g. mechanization, fertilizer, genetic modification of seeds, etc), which raised land productivity. Following Bustos, Caprettini, and Ponticelli (2016), we measure the extent to which locations could take advantage of this higher-intensity cultivation as the differential potential yield under low and high intensity cultivation, according to the FAO-GAEZ agroclimatic model of crop suitability (Fischer, Nachtergaele, Prieler, Van Velthuizen, Verelst, and Wiberg, 2008). Our first IV interacts this differential yield for corn, the dominant crop throughout our period, with a linear time trend.[19]

[18] To construct regions, we draw a box around the continental U.S. (in the Mercator projection) and, beginning from the southwest corner of the box, we overlay squares on top of the box, each of which has an area equal to one tenth of the area of the box. This partitions the continental U.S. into 14 different regions.

[19] To allow for within-location heterogeneity in agroclimatic suitability, we include both the mean differ-

The second major change that we exploit is a shift in world demand that has altered which crops are grown in the U.S., most notably soy.[20] To proxy for which locations saw the greatest gain in (revenue) productivity from this shift, we use the FAO-GAEZ predicted difference in potential yield between soy and wheat (a crop for which demand has remained relatively constant over time) and interact this with a linear time trend.[21] Together, these two sets of supply-equation instruments leverage heterogeneity in geographical exposure to both within-crop and across-crop changes among the three most important food crops for U.S. agriculture.

Finally, when estimating the demand equation (31) we use the climate amenity-based IVs, but additionally control for the agricultural productivity variables (in order to reduce residual variation and the risk that our amenity-based IVs are correlated with unobserved productivity variation). Analogously, our estimation of the supply equation (32) includes controls for the climate amenity variables.

To conclude the three-step procedure we note that, conditional on obtaining consistent estimates of the elasticity parameters (α_1, β_1, α_2, β_2, σ and θ), equations (31) and (32) allow recovery of the geographic fundamentals $\left\{\bar{A}_{it}, \bar{u}_{it}\right\}$ as well. Combined with the earlier estimates of $\{T_{ijt}, M_{ijt}\}$ from step two all model parameters are thereby identified.

3.3 Estimation Results

We begin with estimates from the trade and migration gravity equations in step one, as reported in Table 1. As is standard, our estimate of the elasticity of trade flows with respect to distance, in column 1, is close to minus one: in particular, we obtain $\widehat{\kappa} = -1.35$ (with a standard error of 0.06 based on two-way clustering on origin and destination location).[22] Table 1 also reports estimates of the migration-distance elasticity for each year 1850-2000 in columns 2 through 5. The estimates $\widehat{\lambda}_t$ range from -1.51 to -2.16, with no clear trend over the 150 years spanned.[23]

ential yield for corn and the standard deviation of the differential yield as instruments.

[20]Virtually absent in 1900, soy trailed only corn in terms of both value and acreage in 2000. Roth (2018), for example, argues that much of this rise is due to rising demand for U.S. exports of soy to Asia.

[21]In 1909, wheat was cultivated on 14.7% of harvested acres allocated to principal crops; in 2000, the figure was 17.2%; see USDA (2003). In practice, we use the high- and low-intensity scenarios for soy and wheat, respectively, to reflect the fact that the former was grown predominantly in a more technologically advanced era. As with the first labor supply instrument, we include both the mean soy-wheat differences and the standard deviation of the differences as instruments.

[22]In comparison, Hillberry and Hummels (2008) estimate a distance elasticity equal to -1.31 at a distance of one mile and -0.91 at the mean distance of 523 miles; Dingel (2017) estimates an elasticity of -0.95.

[23]Allen and Arkolakis (2018) calculate the migration distance elasticity for the U.S. for all decennial censuses from 1850-2000 and find a similar similar range (-1.3 to -2.3). Estimated within-country migration distance elasticities are of similar magnitudes in other contexts—for example, -0.7 in Indonesia (Bryan and Morten, 2019) and -1.5 in India (Imbert and Papp, 2020).

Turning to step three, the parameter values implied by our 2SLS estimates of the labor demand equation (31) are reported in Table 2.[24] Column 1 begins with a version that estimates all three parameters (α_1, α_2 and σ) of this equation. It is apparent that these parameters are imprecisely estimated in this specification—the 95% confidence intervals for all three parameters span a wide range of implausible values, such as a negative value for σ and an unreasonably large agglomeration elasticity α_1. It is unsurprising that these estimates suffer from severe multicollinearity given that the regressor $\ln \mathcal{P}_{it}^{1-\sigma}$ is a spatial aggregation with (distance-based) weights that are smooth over space. We therefore, in columns 2 and 3, estimate equation (31) after imposing values of $\sigma = 5$ and $\sigma = 9$ that reflect typical estimates of the trade elasticity (equal to $\sigma - 1$ here).[25] The resulting estimates of α_1 and α_2 are now considerably more precisely estimated, displaying substantial contemporaneous agglomeration spillovers of magnitudes similar to those previously estimated, and more modest historical spillovers.[26] In column 3 (with $\sigma = 9$), we estimate $\widehat{\alpha}_1 = 0.11$ $(SE = 0.034)$ and $\widehat{\alpha}_2 = 0.04$ $(SE = 0.030)$, and treat this as our preferred set of estimates in what follows.[27]

Table 3 displays analogous 2SLS estimates of the parameters in the locational labor supply equation (32), beginning with all four parameters (β_1, β_2, θ and σ) in column 1. For reasons that are similar to the case of Table 2 described above—though now exacerbated due to the presence of *two* market access terms, $\ln \Lambda_{it}^{\theta}$ and $\ln P_{it}^{1-\sigma}$—these estimates also suffer from multicollinearity and hence lack precision. The remaining columns therefore focus on obtaining estimates of β_1 and β_2 under assumed values for θ and σ that relate to prior work; in particular, we consider the four combinations obtained from the same two values of σ as in Table 2 and the values $\theta = 2$ and $\theta = 4$.[28] Regardless of the assumed values, we estimate

[24] In Tables 2 and 3, the reported standard errors are two-way clustered at the location level and at the county-year level. First-stage estimates are presented in Appendix Table C.1 and Appendix Figure C.3 maps the spatial patterns of the predicted change in population from the first-stage regressions.

[25] For example, Eaton and Kortum (2002) estimate a trade elasticity using international trade flows between 3.60 and 12.86 (with a preferred estimate of 8.28) depending on the method, and Simonovska and Waugh (2014) estimate this elasticity to be 4. Donaldson and Hornbeck (2016) estimate a trade elasticity of 8.22 when focusing on intranational trade in the U.S. during the late 19th century. Given the similar setting, we therefore use $\sigma = 9$ as our preferred estimate in what follows.

[26] In reviews of the literature, Rosenthal and Strange (2004) and Combes and Gobillon (2015) conclude that contemporaneous agglomeration elasticities at the city level are likely between 0.03 and 0.08. Roca and Puga (2017) find the combined effect of the immediate productivity gains and the subsequent seven years of experience from moving to a city to imply an agglomeration effect of 0.05. Closer to our setting, by studying how U.S. counties responded to the Tennessee Valley Authority investments, Kline and Moretti (2014) estimate a contemporaneous (but manufacturing sector) agglomeration elasticity of 0.2. And Bleakley and Lin (2012) estimate a long-run agglomeration elasticity (i.e. $\alpha_1 + \alpha_2$) of 0.09 in their study of persistent clustering around portage sites in the U.S..

[27] As reported in Table 2, the minimal first-stage Sanderson and Windmeijer (2016) F-statistic (taken across the two first-stage equations) in this regression is 75.9, indicating that finite-sample 2SLS bias is unlikely.

[28] The closest estimate to the fifty-year bilateral migration elasticity in our model of which we are aware

large positive historical amenity spillovers β_2 consistent with the microfoundations discussed in Section 2.1 where previous durable investments in housing (or other amenity features such as public spaces) increase the amenity value of residing in a location. In contrast, we find that contemporaneous amenity spillovers β_1 are smaller and sometimes negative, albeit imprecisely estimated. In our preferred specification, with $\sigma = 9$ and $\theta = 4$, we estimate $\widehat{\beta}_1 = -0.15\ (SE = 0.279)$ and $\widehat{\beta}_2 = 0.33\ (SE = 0.179)$.[29]

What do these estimated spillovers imply for the degree of persistence and the possibility of path dependence? To answer this, Figure 3 takes our preferred estimates from Tables 2 and 3 and illustrates their position (and their 95% confidence intervals) in the context of the parameter thresholds identified in Propositions 1, 2, and 3. The red star indicates the location of the contemporaneous spillover estimates, $\widehat{\alpha}_1$ and $\widehat{\beta}_1$. From Proposition 1, its location in the yellow region indicates that the dynamic path of the economy is unique—i.e. given any initial distribution of population $\{L_{i0}\}$ and known evolution of geography $\{\tau_{ijt}, \mu_{ijt}, A_{it}, u_{it}\}$, we can uniquely determine (and straightforwardly calculate) the evolution of the economy. However, from Proposition 2, the red star's location near the boundary of uniqueness and non-uniqueness suggests the possibility of very persistent historical shocks. Finally, the green star indicates the location of the estimated combination of contemporaneous and historical spillovers, $\widehat{\alpha}_1 + \widehat{\alpha}_2$ and $\widehat{\beta}_1 + \widehat{\beta}_2$. From Proposition 3, its location in the blue non-unique region suggests the possibility of multiple steady-states of the economy and hence the possibility of path dependence.

4 Does History Matter for the U.S. Spatial Economy?

We have just seen in Figure 3 how the model from Section 2, estimated on U.S. data from 1800-2000 as in Section 3, features unique equilibria, but nevertheless the distinct possibility of both long-lived persistence and genuine path dependence. In this section we proceed to quantify the extent of these phenomena using a decomposition (Section 4.1), a simulation of counterfactual histories (Section 4.2), and an analysis of welfare bounds (Section 4.3).

is due to Monte, Redding, and Rossi-Hansberg (2018), who estimate a location choice elasticity across U.S. counties of 3.30, albeit in a static framework abstracting from bilateral migration costs. We therefore choose the higher value, $\theta = 4$, as our preferred estimate in what follows, but also examine the lower value of $\theta = 2$ in order to approach shorter-horizon estimates (such as that from Caliendo, Dvorkin, and Parro (2019) on U.S. states, with an implied annual elasticity of 0.5).

[29] As reported in Table 2, the minimal (across equations) first-stage Sanderson-Windmeijer F-statistic in this regression is 7.4, indicating the potential for finite-sample 2SLS bias. However, the estimates of β_1 and β_2 based on LIML are both larger in absolute value and lie in the same regions of the parameter space shown in Figure 3 as our preferred estimates.

4.1 How much can history explain U.S. economic geography?

We begin with a model-based decomposition that can be used to quantify the extent to which spatial inequalities across the U.S. exist because of unequal historical conditions. The model identity in equation (20) implies that, for any year T, we can write

$$\ln L_{iT} = C + \underbrace{\phi_0^T \ln L_{i0}}_{\text{history}} + \underbrace{\frac{1}{\gamma}\sum_{s=1}^{T} \phi_1^{T-s} \ln\left(\bar{u}_{is}^{\sigma} \bar{A}_{is}^{\sigma-1}\right)}_{\text{path of fundamentals}} + \underbrace{\frac{1}{\gamma}\sum_{s=1}^{T} \phi_1^{T-s} \ln\left(\Lambda_{is}^{\sigma} P_{is}^{1-2\sigma}\right)}_{\text{path of market access}}, \tag{33}$$

where, as before, $\gamma \equiv 1 + \frac{\sigma}{\theta} - (\alpha_1(\sigma-1) + \beta_1\sigma)$, and now $\phi_1 \equiv \alpha_2(\sigma-1) + \beta_2\sigma$ and $\phi_0 \equiv \phi_1/\gamma$.

This expression allows us to decompose the variation in year-T population levels across locations i into contributions deriving from three terms involving variation in: (i) exogenous historical conditions L_{i0} at any "initial" date (i.e. $t = 0$) in the past; (ii) paths of exogenous productivities $\bar{A}_{it}$ and amenities $\bar{u}_{it}$ in these locations; and (iii) paths of endogenous market access for trade P_{it} and migration Λ_{it}. While it is clear that term (i) reflects "history" and term (ii) reflects geography "fundamentals", term (iii) involves a combination of history and fundamentals that does not linearly decompose further. We therefore view term (i) as a lower bound on the role of history in this decomposition.

Panel (a) of Figure 4 presents the results of this exercise. We set T to be the year 2000 and consider initial years $t = 0$ ranging from 1800-1950. In each case we report the variance (across the N locations) of each of the three terms in equation (33) divided by the variance of $\ln L_{iT}$. History plays a large role. For example, historical conditions in 1800 account for at least 32.6% of the variation in (log) population in 2000, rising to 56.6% in 1850 and 63.9% in 1900. To put these magnitudes in context, we note that equation (33) is the $(T-1)$-period iterated version of an AR(1) process with parameter ϕ_0, augmented by the spatial interactions arising in the market access term. Our parameter estimates imply that $\widehat{\phi_0} = 0.90$, which is indicative of slow convergence purely because of the logic of agglomeration (since ϕ_0 is a function of the spillover elasticities and the trade and migration elasticities, not migration costs).

We repeat this calculation using an analogous expression for ex-post welfare W_{it} in Panel (b) of Figure 4.[30] While history matters for the distribution of welfare, it matters much

[30] An analagous derivation to equation (33) relying on equation (11) yields the following expression:

$$\ln W_{iT} = C - \underbrace{\frac{\phi_0^T}{\theta} \ln L_{i0}}_{\text{history}} - \underbrace{\frac{1}{\theta\gamma}\sum_{s=1}^{T} \phi_1^{T-s} \ln\left(\bar{u}_{is}^{\sigma} \bar{A}_{is}^{\sigma-1}\right)}_{\text{path of fundamentals}} + \underbrace{\ln \Lambda_{iT} - \frac{1}{\theta\gamma}\sum_{s=1}^{T} \phi_1^{T-s} \ln\left(\Lambda_{is}^{\sigma} P_{is}^{1-2\sigma}\right)}_{\text{path of market access}}.$$

less than it does for the distribution of population: the lower bound on the extent to which historical conditions in 1800 account for variation in (log) ex-post welfare in 2000 is 4.0%, though this rises to 26.5% and 54.7% in the case of the more recent historical conditions in 1850 and 1900, respectively. That history seems to matter more for the distribution of where people live than for their welfare—a consistent theme of our analysis in this section—suggests that historical differences in welfare are partially arbitraged away both contemporaneously through the movement of goods and over time by the movement of people.

4.2 Persistence in the U.S. spatial economy

The results in Section 4.1 suggest that unequal historical conditions play a large role in accounting for spatial inequalities in the U.S. economy today. But a decomposition exercise like this one is unable to answer questions about the causal impact of historical conditions—how different would the spatial economy look today if historical conditions had been different?

To answer this question, we need to compare actual historical conditions to counterfactual alternatives. One could imagine many counterfactual histories of interest, but we seek particular inspiration from the vagaries of industrial success that struck America's communities at the turn of the 20th century. This period—known variously as the Technological Revolution or the Second Industrial Revolution—was a period of rapid productivity growth across a number of different industries due to wide-spread adoption of technological innovations such as the internal combustion engine and electrification.[31]

Crucially for us, the adoption of these innovations varied across locations within the United States, often for reasons that (in hindsight) can be partially attributed to historical "luck". For example, Detroit's rise as the "Motor City" plausibly owes something to the fact that Henry Ford happened to be born on a nearby farm. Or perhaps Buffalo became the "City of Light", in more than just a name, because it was chosen to host the Pan-American Exposition at a time (1901) when Thomas Edison desired to show off his newly invented AC power by adorning Buffalo's Exposition buildings with light bulbs.[32]

Examples like these illustrate how relatively similar locations may (or may not) have been the fortunate recipients of positive productivity shocks in a time of technological change. To study the consequence of such hypotheticals, we generate a set of counterfactual histories in which productivity fundamentals are randomly swapped between pairs of similar locations.

Note that a similar decomposition is not possible for ex-ante welfare $\Omega_{i,t}$, as from equation (14) it can be written solely as a function of inward and outward market access.

[31] See e.g. Landes (2003).

[32] Unfortunately, the Exposition's interiors were less well-lit. To illuminate the operating table used (unsuccessfully) to remove a bullet from President McKinley, who was shot at the event, the doctor had to rely on a mirror to reflect the rays of the setting sun (Leech, 1959, p. 596). He died of gangrene a week later.

For example, what if Cincinnati (with a population of 330,000 in 1900) had been chosen instead of Buffalo (population 350,000 in 1900) for the site of the Pan-American Exposition?

To operationalize this idea, albeit in an abstract manner, we carry out a set of simulations, each indexed by b, as follows. First, we rank all locations in terms of their observed $L_{i,1900}$. Second, we form pairs p of locations based on their nearest neighbor in this ranked distribution, starting at the top; for example, Erie County, NY (home to Buffalo) and Hamilton County, OH (home to Cincinnati) occupy ranks 11 and 12 in the distribution.[33] Third, in simulation b we draw (independently) for each pair p random variable $W_p^{(b)}$ that is distributed Bernoulli(1/2). When $W_p^{(b)} = 1$, we swap the values of the fundamental productivity in 1900 $\overline{A}_{i,1900}$ among the two locations within pair p; and when $W_p^{(b)} = 0$ we leave the pair unchanged. Fourth, we then simulate the model forwards from 1900 onward holding fixed all other exogenous locational characteristics in the model (i.e. $L_{i,1850}, \overline{u}_{i,1900}$, and the entire path of $\overline{A}_{it}$ and $\overline{u}_{it}$ for $t > 1900$) at their values estimated in Section 3.3 (and with $\overline{A}_{it} = \overline{A}_{i,2000}$ and $\overline{u}_{i,t} = \overline{u}_{i,2000}$ held fixed for all $t > 2000$). This generates a stream of counterfactual predictions for all the endogenous variables in the model (which we denote as $L_{it}^{(b)}, W_{it}^{(b)}, \Omega_{it}^{(b)}$, etc.) at all dates $t \geq 1900$.

We then repeat these four steps for all $B = 100$ simulations. We will also draw on an additional $(B+1)^{th}$ simulation (the output of which we label as, for example, $L_{it}^{(F)}$, for "factual") in which there are no swaps at all. This corresponds, for $t \leq 2000$, to the factual path taken by variables such as L_{it} in the data. For years $t > 2000$ this exercise therefore simulates forward a model that (by design) fits the past data perfectly. We simulate each counterfactual history (and the factual history) forward 30 generations past the year 2000, to the year 3500. While these projections are obviously heroic, such a long horizon turns out to be necessary in order to capture the full scope of historical persistence in this model.

To summarize, each of these "swap" counterfactual history simulations holds everything in the model constant apart from the fundamental sources of productivity in 1900, $\overline{A}_{i,1900}$, and even the $\overline{A}_{i,1900}$ *distribution* is held exactly constant (not just on aggregate but also across the $N/2$ pairs of locations). The only thing being perturbed in any counterfactual history is the within-pair assignment of productivity in 1900 among pairs of locations that are as close as possible to one another in terms of their 1900 populations.

[33]Other examples of pairs include the counties that are home to Worcester and San Francisco (ranks 17 and 18), Providence and Baltimore (ranks 21 and 22), New Haven and New Orleans (ranks 23 and 24), and Louisville and Minneapolis (ranks 27 and 28). Due to the odd number of locations, that with the smallest 1900 population—a subset of Craig County, VA—is without a partner. We therefore leave its productivity unchanged in every simulation.

4.2.1 Local persistence elasticities

We begin by quantifying the local impact of these swap counterfactuals. While the counterfactual changes are to productivities $\overline{A}_{i,1900}$, a useful way to summarize the effect of these shocks can be obtained by noting that, from the perspective of any year $t > 1900$, the only impact of these shocks is to alter $\{L_{i,1900}\}$, the initial conditions of the model's only state variable. We therefore study the impact of a change in such initial conditions, rather than the underlying shocks to $\overline{A}_{i,1900}$ that altered these initial conditions, as follows.

For any generic "outcome" of interest, O_{it}, we use the data generated by the model simulations in order to estimate the regression

$$\ln O_{it}^{(b)} = \delta_{it}^{O} + \eta_{it}^{O} \ln L_{i,1900}^{(b)} + \varepsilon_{it}^{O(b)} \tag{34}$$

separately for each location i and time period $t > 1900$. Our interest lies in the *local persistence elasticity* (for outcome O), denoted by η_{it}^{O}. This elasticity measures the average relationship, across the B simulations, in location i between that location's historical population $L_{i,1900}^{(b)}$ and its value for the outcome $O_{it}^{(b)}$ in some later period t. The error term $\varepsilon_{it}^{O(b)}$ in equation (34) is almost surely correlated with $L_{i,1900}^{(b)}$, for any outcome—as, for example, equation (20) makes clear when O_{it} represents population. But $\ln \overline{A}_{i,1900}^{(b)}$ can serve as an excludable IV for consistent estimation of η_{it}^{O} given that it is randomly assigned (by design).

Figure 5 reports the distribution (across locations) of the estimated values of the local persistence elasticity for population $\widehat{\eta}_{it}^{L}$ that corresponds to each of the years $t = 1950\text{-}3500$.[34] For example, the median elasticity $\widehat{\eta}_{i,2000}^{L}$, 100 years after our simulated shocks, is 0.89, and it remains high at 0.45 after 500 years. While there is considerable heterogeneity across locations, even the lowest inter-quartile range value is 0.80 at $t = 2000$. This suggests that in a dynamic economic geography model like the one developed here, it should be considered the norm, rather than the exception, to observe that an event that raises a location's population at a point in time leads to centuries-long economic persistence.

Also shown in Figure 5 is the distribution of the elasticities $\widehat{\eta}_{it}^{W}$, where the outcome O_{it} is ex-post welfare $W_{it}^{(b)}$. As expected, the local welfare elasticities $\widehat{\eta}_{it}^{W}$ are considerably lower than the local population elasticities $\widehat{\eta}_{it}^{L}$—because of both trade and migration, the welfare of a location draws on both local and nearby geographical advantage, so the local impact of local shocks is muted by spatial interactions and arbitrage. Still, the median elasticity after 100 years is 0.21 and it is not much lower, at 0.11, after 500 years.

[34] To account for the possibility that the elasticities $\widehat{\eta}_{it}^{O}$ are estimated with error, in this figure we weight each location by the inverse of the square of the estimated standard error of its estimate. The instruments are typically very strong, with a mean (median) first stage F-statistic of 492 (166) and 85% of locations' F-statistics exceeding 10.

Finally, Figure 5 also reports the corresponding elasticities for ex-ante welfare, Ω_{it}. Even one period after the shock, the local elasticities $\widehat{\eta}^{\Omega}_{it}$ are all essentially zero. This means that while local shocks cast a long shadow over many characteristics (populations, prices, wages, productivity, amenities) of the local economy, they have little bearing on the welfare of children who are born in such locations because those children always have the option to migrate outwards. However, while local shocks have no impact on the *spatial distribution* of the next generation's welfare, this is not to say that such shocks have no *aggregate* welfare consequences; indeed, as we will see below, we find precisely the opposite.

4.2.2 Fragility and resilience

As we have seen, local productivity swaps leave a sizable impact on their local economies, even many centuries after the shocks occurred. One consequence of this is that we should expect variability of local outcomes across simulations. But which locations tend to be fragile in the face of these shocks and which tend to be resilient? To explore this for the year 2000, Figure 6 shows a plot of the standard deviation of $\ln L^{(b)}_{i,2000}$ (as well as $\ln W^{(b)}_{i,2000}$ and $\ln \Omega^{(b)}_{i,2000}$) across the B simulations against the factual population in 2000, $L^{(F)}_{i,2000}$. We see that no location is immune to these shocks (the minimum standard deviation of log 2000 population is 1.12), not even those in the largest locations (the average standard deviation among the year 2000's 25 largest locations is 1.99). And many smaller locations see extremely large variability across our simulations.[35]

As expected, similar patterns are on display for the two welfare measures, $\ln W^{(b)}_{i,2000}$ and $\ln \Omega^{(b)}_{i,2000}$, but in a substantially dampened fashion (especially for Ω_{it}) throughout the distribution. However, even ex-ante welfare is hardly impervious to 100 year-old shocks; for example, the largest 25 locations have an average standard deviation of $\ln \Omega^{(b)}_{i,2000}$ of 0.25.

4.2.3 Spatial configuration

Having seen that local historical shocks leave long-lasting impacts on individual locations, a natural question is whether these impacts can be so large as to re-orient the spatial configuration of the entire economy.

To investigate this, we examine the correlation of the entire distribution of population across counterfactual historical simulations. For any pair of simulations, b and b', and any year t, we calculate $\chi^{L}_{bb',t} \equiv corr(\ln L^{(b)}_{it}, \ln L^{(b')}_{it})$ as a way of asking how similar the spatial distribution of (log) population is across these two simulations. We do this using all pairs of

[35] This tendency for smaller locations to have relatively greater variability across simulations, however, does not account for the cross-location heterogeneity of $\widehat{\eta}^{L}_{i,2000}$ seen in Figure 5; the correlation between $\widehat{\eta}^{L}_{i,2000}$ and $\ln L^{(F)}_{i,2000}$ is -0.06.

the 100 simulations of counterfactual history, and also that corresponding to actual history (i.e. for $\ln L_{it}^{(F)}$), so there are $5,050$ (i.e. $\frac{101\times100}{2}$) values of $\chi_{bb',t}^{L}$ for each year.

Figure 7 reports the interquartile spread of this statistic across those $5,050$ values. Prior to the onset of the shocks in 1850 this correlation (i.e. $\chi_{bb',1850}^{L}$) is equal to one, of course. Upon the realization of the shocks in 1900, it declines to 0.79 and continues to fall, reaching 0.76 by the year 2000 and 0.70 in the year 3500. At the same time, the spread of correlations across simulation pairs rises, slowly, such that by the year 3500, 25% of simulation pairs have a correlation less than 0.52, whereas 25% of simulation pairs have a correlation greater than 0.91. A similar pattern is seen in Figure 7 for the cross-simulation correlation of ex-post and ex-ante welfare, $\chi_{bb',t}^{W}$ and $\chi_{bb',t}^{\Omega}$.[36]

Summarizing, while Figure 5 documents that historical shocks can have extremely persistent effects on the local economy, Figure 7 points to the possibility of something stronger: long-run outcomes that remain permanently distinct across simulations. This is indicative of path dependence, as we now discuss.

4.3 Path Dependence in the U.S. Spatial Economy

Following Section 2.3, we now focus on the behavior of locations' ex-ante welfare, Ω_{it}, because they will satisfy $\Omega_{it} = \Omega$ once any steady-state has been reached. Figure 8 plots the path of population-weighted $\ln \Omega_{it}^{(b)}$ over time for each simulation b (the various colors used to illustrate these paths reflect the spatial orientations underlying them, as we discuss shortly), as well as that for the projected factual path $\ln \Omega_{it}^{(F)}$ in light green.[37] Recall that all of our 101 simulated economies feature exactly the same paths of fundamentals from $t = 1950$ onward, so all heterogeneity across these simulations after 1900 is purely the result of heterogeneity in the state variable $\{L_{i,1900}^{(b)}\}$ induced by the simulated productivity swaps in 1900.

Several features of Figure 8 stand out. First, aggregate welfare in each simulation tends to rise as the pursuit of more attractive locations by mobile children entering adulthood pushes the population distribution closer and closer to a more efficient allocation of resources (even though it is certainly not guaranteed to do so). Second, this process stabilizes in most simulations over the first 300-400 years, with relatively small adjustments over the 3,000 years that follow. Third, these small ensuing adjustments do lead to genuine steady-states in a

[36]Unlike population and ex-post welfare, the median ex-ante welfare correlations $\chi_{bb',t}^{\Omega}$ do increase from their nadir of 0.20 in 2200 to 0.58 in 3500; this increase is due in part to the fact that ex-ante welfare is equalized across all locations in the steady-state and we set the correlation between any pair in which at least one is in the steady-state to one (as it would be otherwise undefined).

[37]As our identification procedure recovers productivity and amenity levels to scale, to compare recovered geographies across years, we choose the scale such that the factual welfare in years 1900 and 1950 have the same mean as in the year 2000. No choice of scale is necessary for subsequent years, as the geography remains constant at its year 2000 values.

number of cases (illustrated with yellow dots) but in many other cases the small adjustments are still occurring, albeit at an exceedingly slow pace.[38] Fourth, even though there is broad convergence of many paths to the same or similar steady-states, this convergence is by no means uniform, and periods of temporary *divergence* are common. Fifth, many of these simulations stabilize at differing levels of aggregate welfare, which implies that this model displays not only path dependence but path dependence with aggregate welfare consequences. Finally, these aggregate welfare consequences can be substantial. For example, the welfare gap across the three steady-states shown is 1.29 log integers—which, for context, is equivalent to 516 years of growth at 0.25% per annum. And while the factual economy ends up closer (in welfare terms) to the best steady-state than to the worst by the year 3500, it is still poorer than the best steady-state by an amount that corresponds to 181 years of foregone 0.25% annual growth.

Put together, these findings imply that the relatively small swap counterfactual shocks in our simulations are large enough to tip the model economy towards different steady-states, implying that the basins of attraction of these steady-states are also relatively small. One might therefore conjecture that larger shocks in 1900 could have led to even greater divergence in aggregate welfare. This is where the analytical bounds on $\Omega(\{L_{i0}\})$ obtained in Proposition 4 prove useful. Figure 8 also plots the upper bound $\bar{\Omega}$ according to equation (22) as a dashed red line. While none of our simulations reaches this upper bound, the cross-simulation variation is considerably larger than the gap between the best steady-state and $\bar{\Omega}$.

However, the lower bound $\underline{\Omega}$ from equation (23), which is calculated in Table 4, implies that we cannot rule out the presence of alternative 1900 population distributions that would have led this same post-1950 economy towards considerably (many orders of magnitude) lower aggregate welfare. The factual 1900 distribution of population $\{L_{i,1900}\}$ was hence potentially quite close, in this welfare sense, to the basins of attraction of relatively good steady-states, when compared to what might have been. This is consistent with the finding in Section 2.4 that the basins of attraction of good steady-states tend to be larger than those of dominated steady-states.

Table 4 also reports the five components of the upper and lower bounds, $\bar{\Omega}$ and $\underline{\Omega}$, as derived in Proposition 4. All of these components allow substantial latitude for historical conditions to matter because their individual contributions to the ratio of upper-to-lower bounds are each large. For example, one might have conjectured that the mere presence

[38] For us to classify a simulation as being in a steady-state, we require that Ω_{it} is equalized across all locations to numerical precision. By this definition, 11 of 101 simulations have reached a steady-state by the year 3500. Of the remaining 90 simulations, the median standard deviation of $\ln \Omega_{it}$ was 0.09, with an interquartile range of 0.05 and 0.15, and none of the 90 have a standard deviation of $\ln \Omega_{it}$ less than 0.001.

of large agglomeration spillover elasticities in each location would mean that the aggregate welfare differences coming from concentration *per se* would be larger than any effects arising from where that concentration occurs. But this conjecture turns out to be incomplete since the effects of concentrating in superior locations (with better migration costs, trade costs, or geographic fundamentals shown in columns 2-4, respectively) are similar in magnitude to that from overall scale (in column 5).

Finally, the presence of multiple steady-states begs the question of what these various configurations look like, geographically speaking. As shown in the maps of Figure 9, our 100 simulations converge upon seven different forms of spatial organization by 3500, and the factual economy simulation converges upon an eighth.[39] The eight outcomes differ in many senses, such as: where the location with the largest population is located, the relative size of the largest location and secondary location(s) of agglomeration, the importance of the West relative to the East, the extent to which historically outsized concentrations (such as the Northeast) still hold sway in 1,500 years, and (as per Figure 8) aggregate welfare.[40] The highest-welfare steady-state (map g) has one dominant location (in Charlottesville, Virginia) but the second-best configuration (map f), which is the factual history simulation, also has substantial population concentration in Charlottesville, but a larger concentration in Denver, Colorado. On the other hand, as illustrated in the scatter plots of Figure 9, all eight spatial configurations display a similar degree of correlation, on average, between their year 3500 simulated populations and the factual population in 2000.

4.4 Summary

The results above paint a nuanced picture of how we might expect history to matter in a dynamic economic geography model when it is estimated to fit long-run U.S. data. We have seen how merely swapping the productivity fundamentals of similarly-sized locations in 1900—while holding fixed all other exogenous features before, during and after the year 1900—can set in motion a wide range of endogenous consequences, playing out on multiple spatial scales.

On one end of the scale, there is a great deal of local persistence: local shocks have large effects on their local economies, and these effects continue to leave their trace on local outcomes over many centuries. At the same time, these local shocks also have large consequences for the spatial configuration of the economy as a whole because of the logic of path

[39] Specifically, we define a "form" here on the basis of the identity of the location with the greatest concentration of population. Simulations within the same form may nonetheless differ in the extent to which population concentrates in that location and/or the distribution of economic activity elsewhere.

[40] The color of the dots in the scatter plots beneath each map correspond with the colors of the welfare paths shown in Figure 8.

dependence: the 101 random draws of swaps that we study converge towards at least eight distinct configurations, each of which features unique patterns of economic agglomeration and substantial differences in aggregate welfare.

In short, history definitely matters in our estimated model U.S. economy. It matters in terms of slow convergence to any steady-state, in terms of where a steady-state agglomerates spatially, and even in terms of whether relatively good or bad steady-states are selected by historical conditions. And as the analytical bounds of Table 4 suggest, it is likely that our swap simulations are only probing a small extent to which history can matter in this context.

5 Conclusion

It is not hard to look at the geographic patterns of economic activity around us and believe both that agglomeration forces are at work and that they may even be strong enough to cause a self-reinforcing clustering of economic activity. This opens up the possibility that there are many such spatial configurations in which mobile factors could settle—some good, some bad—as well as the potential for historical accidents, such as initial conditions or long-defunct technological shocks, to play a long-lived or even permanent role in determining the distribution and efficiency of spatial allocations.

This paper has sought to develop a dynamic economic geography framework that can be used to characterize and quantify these possibilities. We have derived conditions on how contemporaneous and historical agglomeration externalities in production and amenities govern: (i) the existence and uniqueness of equilibria; (ii) the duration of persistence of shocks around a steady-state; (iii) the scope for multiple steady-states; and (iv) the extent to which such multiple steady-states may deliver different amounts of aggregate welfare.

A particularly rich region of the model's parameter space—and one that is consistent with our estimates based on U.S. Census data from 1800 onwards—is where equilibria are unique and easy to solve for, persistence lasts many centuries, and only minor perturbations in historical conditions can lead the economy towards distinct steady-states with substantial differences in overall efficiency. One implication of this parameter region is that temporary economic events in many domains will leave large and long-lived geographical traces, consistent with the numerous empirical studies discussed in the Introduction. Answers to questions about optimal place-based policy will also be particularly subtle in the presence of such features.

While we have developed this paper's empirical and theoretical tools in the hopes of an improved understanding of regional economic geography, these techniques could be applied to other settings in which increasing returns and coordination failures, and hence multiplicity and path dependence, have long appeared as objects of theoretical interest that lack a corresponding amount of high-dimensional quantification and simulation. Potential areas of application could include: intra-city geographical phenomena such as residential segregation, sorting, and so-called "tipping" dynamics (Schelling, 1971; Card, Mas, and Rothstein, 2008; and Lee and Lin, 2018); traditional "big push" models of economic development (Rosenstein-Rodan, 1943; Murphy, Shleifer, and Vishny, 1989; and Krugman and Venables, 1995); competition policy questions surrounding technology adoption in the presence of network effects and switching costs (David, 1985; and Farrell and Klemperer, 2007); and the study of dynamic questions of political economy such as those surveyed in Acemoglu and Robinson (2005).

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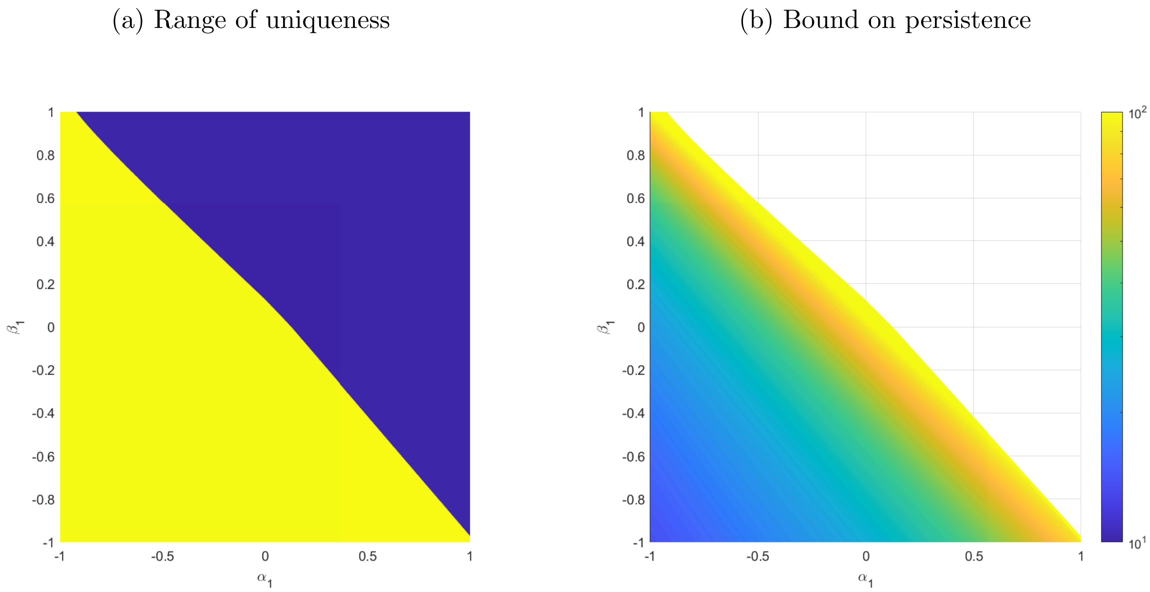
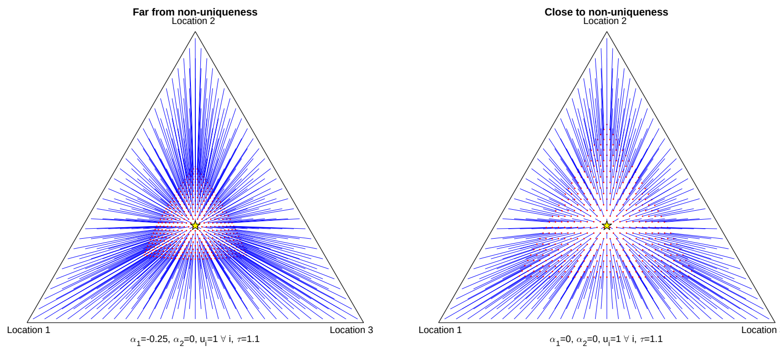
Figure 1: Illustration of Propositions 1 and 2



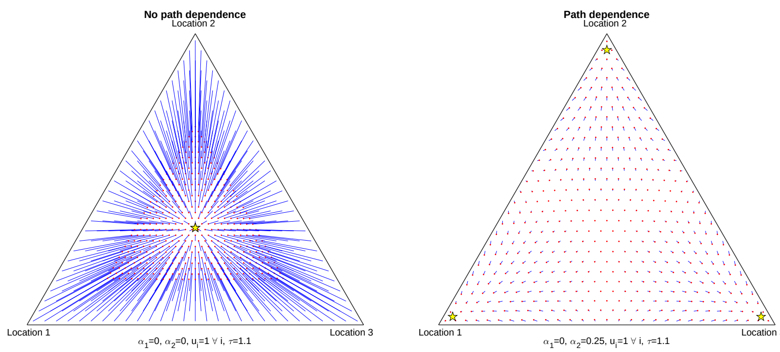
Notes: This figure illustrates the regions of the parameter range (in the space of α_1 and β_1, holding other parameters constant) relating to Propositions 1 and 2. The left panel shows (in yellow) the range of α_1 and β_1 that satisfy the conditions for the uniqueness of equilibrium, as per Proposition 1, holding trade and migration elasticities constant at the values $\sigma = 9$ and $\theta = 4$. The right panel shows the upper bound on the persistence of the economy, as measured by the largest eigenvalue of $|\mathbf{B}^{-1}|\left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1, \beta_1\right)\right)^{-1}\mathbf{C}\,|\mathbf{B}|$, as per Proposition 2, holding trade and migration elasticities constant at the same values and choosing $\alpha_2 = 0.04$ and $\beta_2 = 0.33$ to be consistent with the estimation results below. Note that the bound on persistence approaches infinity as the parameter constellation approaches the threshold where the sufficient condition for uniqueness is no longer satisfied (but we top-code the color scale for readability).

Figure 2: Illustrating the implications of Propositions 2-4 in a three-location economy

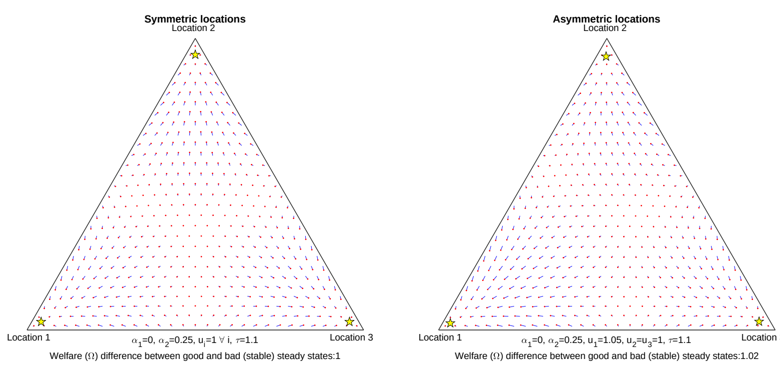
(a) Prop. 2: The closer to non-uniqueness, the greater the persistence



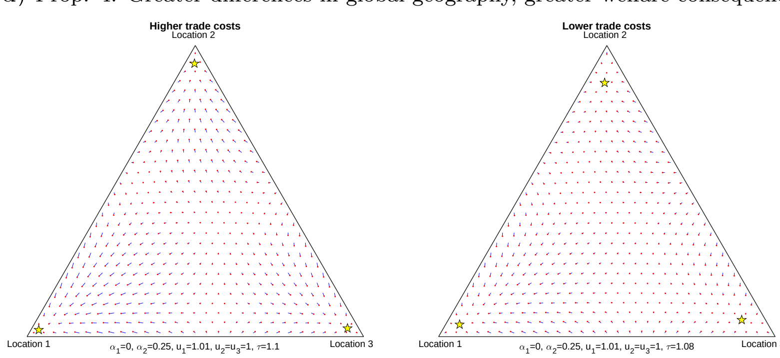
(b) Prop. 3: Historical spillovers can lead to path dependence



(c) Prop. 4: Greater differences in local geography, greater welfare consequences

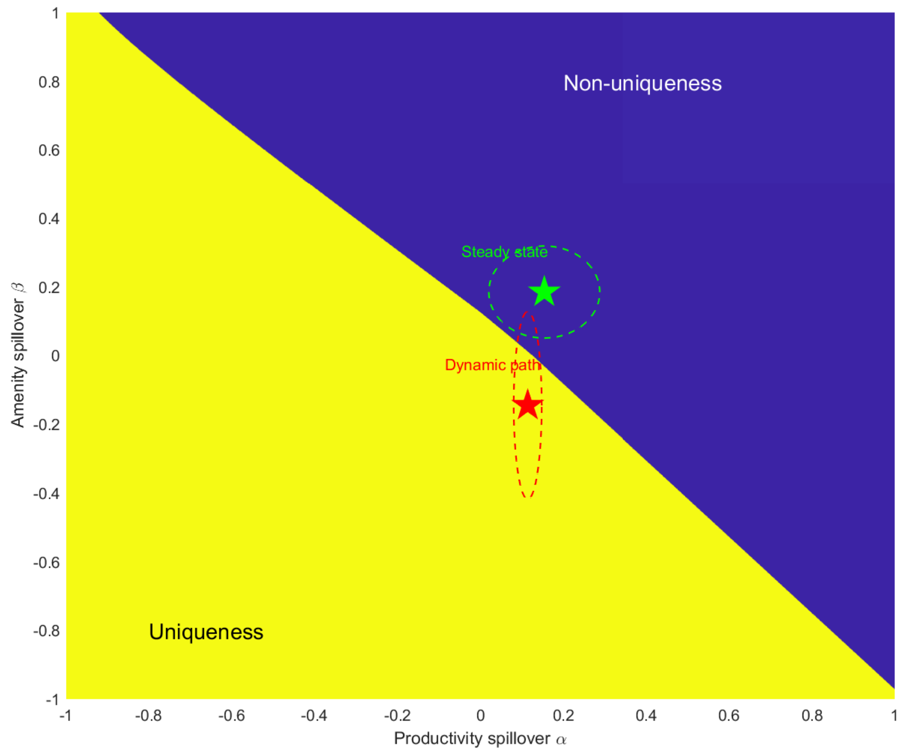


(d) Prop. 4: Greater differences in global geography, greater welfare consequences



Notes: These figures illustrate phase diagrams for a three-location example economy. Arrows indicate the change in the equilibrium distribution of population from one period to the next, with yellow stars denoting stable steady-states. See Section 2.4 for details.

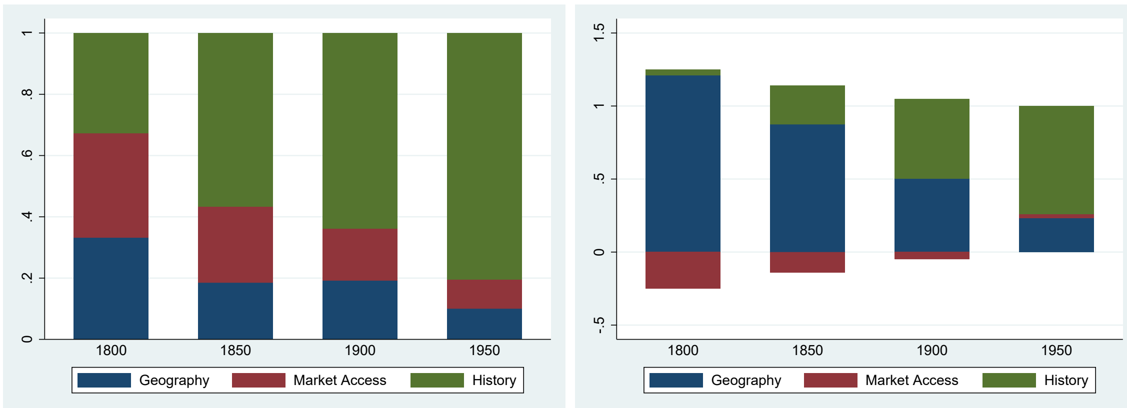
Figure 3: Agglomeration spillover parameter estimates



Notes: This figure illustrates the parameter estimates (holding σ and θ constant) obtained in Section 3.3. The red star denotes $\widehat{\alpha}_1$ and $\widehat{\beta}_1$, which lies in the yellow region of equilibrium uniqueness following Proposition 1. The green star denotes $\widehat{\alpha}_1 + \widehat{\alpha}_2$ and $\widehat{\beta}_1 + \widehat{\beta}_2$, which lies in the blue region, indicating the possibility of multiple steady-states following Proposition 2. Confidence intervals are shown with dashed lines.

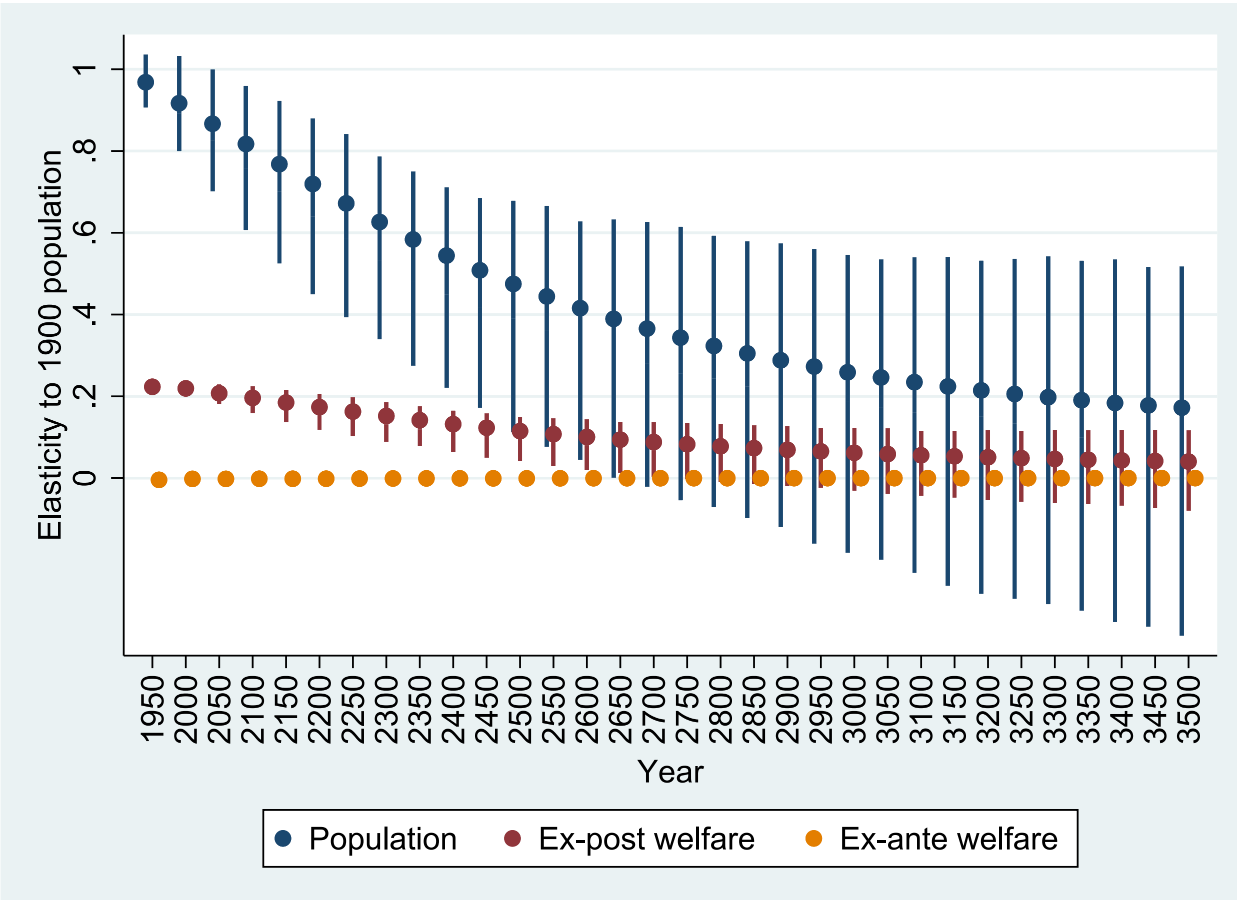
Figure 4: How much of the spatial distribution of economic activity today is due to history?

(a) Population ($L_{i,2000}$) (b) Ex-post welfare ($W_{i,2000}$)



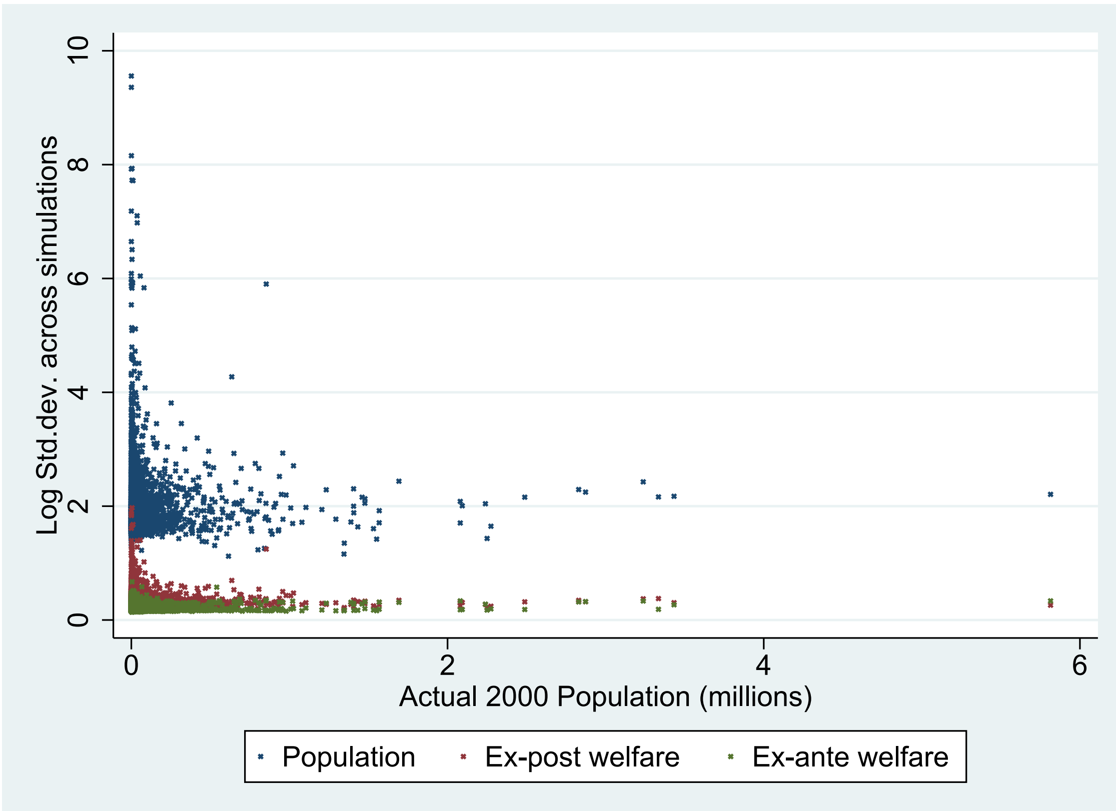
Notes: This figure presents the variance decomposition of the observed spatial distribution of economic activity in the year 2000 into three components, as per equation (33): geography fundamentals (i.e. the complete history of realizations of productivities $\bar{A}_{it}$ and amenities $\bar{u}_{it}$ from $t = 0$ until the present), market access (i.e. the complete history of goods market access P_{it} and labor market access Λ_{it} from $t = 0$ until the present), and history (i.e. the population distribution in $t = 0$, L_{i0}). The decompositions shown correspond to four choices of initial year $t = 0$: 1800, 1850, 1900, and 1950. Panel (a) presents the decomposition for the observed distribution of population in the year 2000 ($L_{i,2000}$), and panel (b) presents the equivalent for ex-post welfare ($W_{i,2000}$).

Figure 5: How persistent are local historical shocks for the local economy?



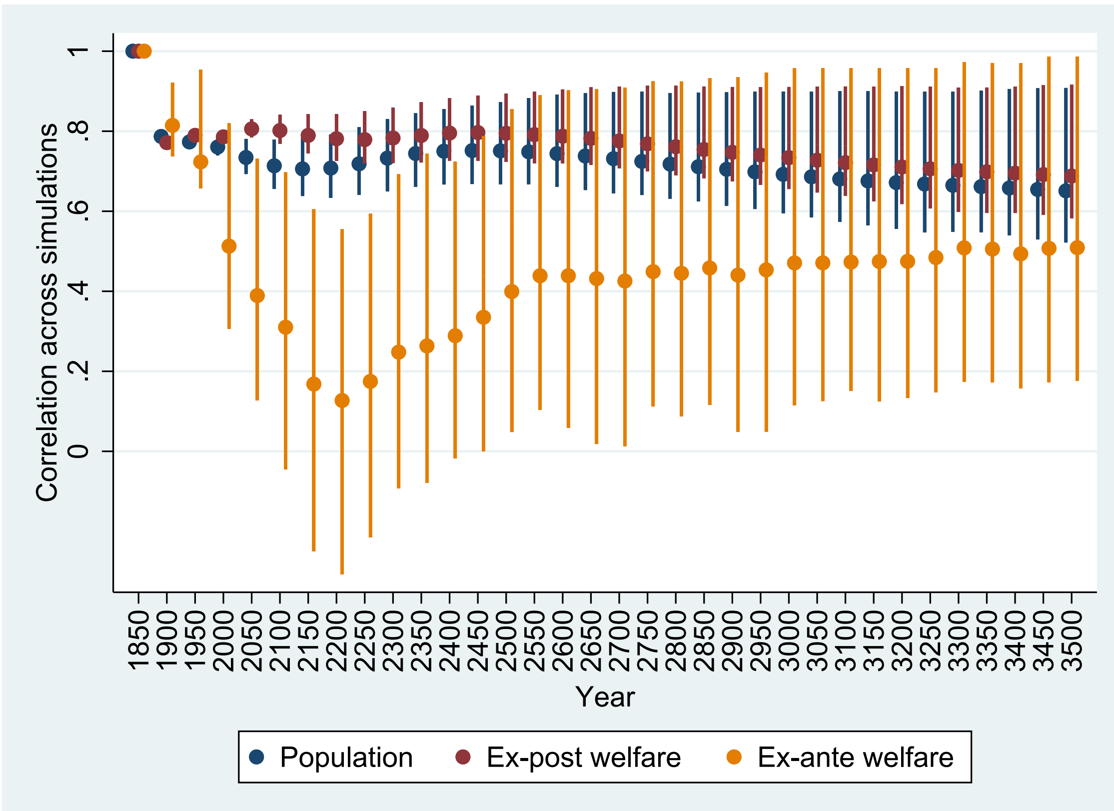
Notes: This figure shows the distribution of estimated elasticities of the local persistence elasticity, $\widehat{\eta}_{it}^{O}$ for various outcomes "O" (population $L_{i,t}$, ex-post welfare W_{it}, and ex-ante welfare Ω_{it}), across all locations i and for each indicated year t. Following equation (34), η_{it}^{O} is obtained in a regression of $\ln O_{it}^{(b)}$ on $L_{i,1900}^{(b)}$ across 100 simulations b, separately by location-year, using the (randomly assigned) value of exogenous productivity $\overline{A}_{i,1900}^{(b)}$ as an IV. Each simulated history randomly shuffles the realized exogenous productivity in the year 1900 between all pairs of locations, where pairs are assigned to locations with the closest 1900 populations. The dots indicate the mean estimated elasticity $\widehat{\eta}_{it}^{O}$ across all locations (and the bar indicates the interquartile range) in a given year, weighting elasticity estimates by the inverse of the square the estimate's standard error.

Figure 6: How resilient are locations to historical shocks?



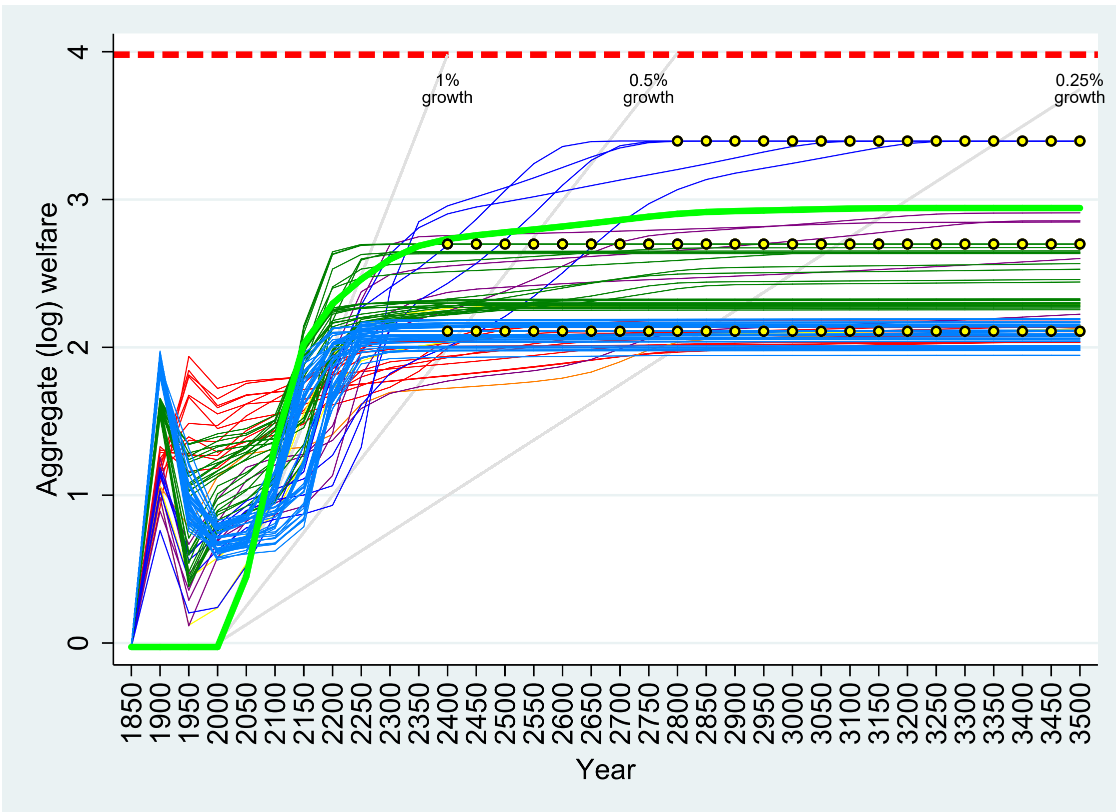
Notes: This figure plots the standard deviation of log population $\ln L_{i,2000}^{(b)}$, log ex-post welfare $\ln W_{i,2000}^{(b)}$, and log ex-ante welfare $\ln \Omega_{i,2000}^{(b)}$ in the year 2000 across 100 different simulations b of history against each location's actual year 2000 population $L_{i,2000}$. Each simulated history randomly shuffles the realized exogenous productivity $\overline{A}_{i,1900}^{(b)}$ in the year 1900 between all pairs of locations, where pairs are assigned to locations with the closest 1900 populations.

Figure 7: How persistent are historical shocks for the economy as a whole?



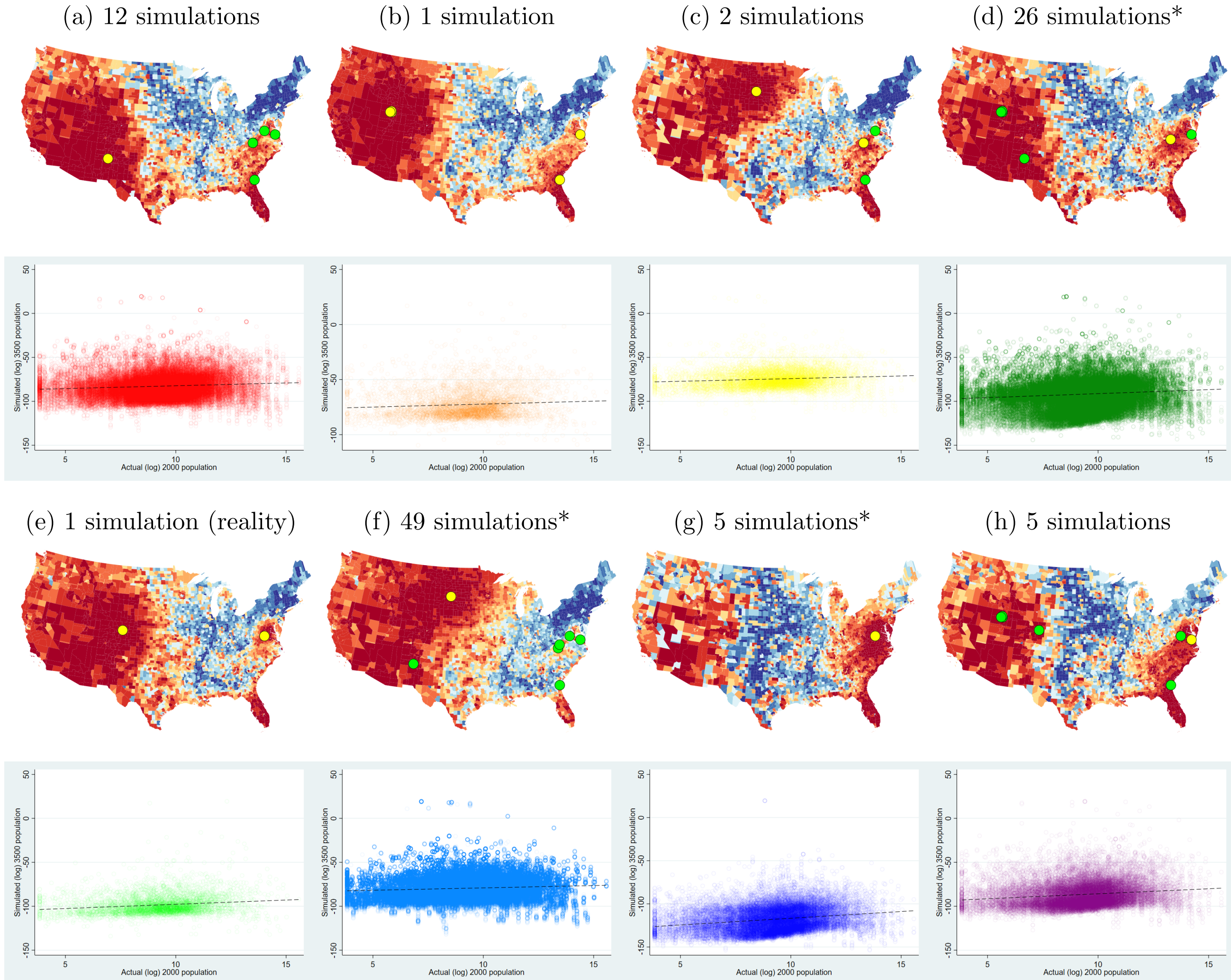
Notes: This figure shows the correlation across simulations in the location of economic activity over time. For each year and each possible pair of simulations, we calculate the cross-location correlation in log population $\ln L_{i,t}^{(b)}$, ex-post log welfare $\ln W_{i,t}^{(b)}$, and ex-ante log welfare $\ln \Omega_{i,t}^{(b)}$. The dot indicates the mean correlation across all possible pairs within a year and the bar reports the interquartile range. We set the correlation of ex-ante log welfare equal to one if either simulation in a given year is in a steady-state (since steady-state ex-ante welfare is equalized across locations). As there are 100 simulated histories (plus the observed factual history), there are 5,050 possible correlations ($101 \times 100 \times \frac{1}{2}$) in each year. Each simulated history randomly shuffles the realized exogenous productivity $\overline{A}_{i,1900}^{(b)}$ in the year 1900 between all pairs of locations, where pairs are assigned to locations with the closest 1900 populations.

Figure 8: How does history affect aggregate welfare?



Notes: This figure plots the evolution of the population-weighted distribution of ex-ante welfare, $\ln \Omega_t^{(b)} \equiv \sum_{i=1}^{N} \left(\frac{L_{it}^{(b)}}{\bar{L}} \right) \ln \Omega_{it}^{(b)}$, over time for 100 different simulations b of history. The color of each path corresponds to the location with the greatest concentration of population in the year 3500 (see Figure 9). Also shown (in lime green) is the path of aggregate welfare for the factual economy (normalized to its 2000 value because the scale of aggregate welfare is unidentified for years prior to 2000). Yellow circles indicate when a steady-state has been reached. The red dashed bar indicates the maximum possible steady-state aggregate welfare according to Proposition 4. Grey lines indicate annualized growth rates from the year 2000.

Figure 9: Possible distributions of economic activity in the year 3500



Notes: This figure partitions 100 simulations into 7 groups (plus one more, for the simulation of the factual economy, shown in panel e) based on the location of greatest population in 3500. In each map, locations are colored on the basis of their population relative to the distribution, with red (blue) indicating more (less) population. Yellow dots denote a location at which all (and for green dots, some) of the simulations featured more than 10 million persons (in total 2000 population level units) concentrated in the location. Underneath each map, the corresponding scatter plot relates $\ln L_{i,3000}^{(b)}$ to $\ln L_{i,2000}$ across all of the locations and simulations b in the group, with the scatter-point colors corresponding to the paths in Figure 8. The number of simulations in each group is indicated above each map and groups that include a steady-state by 3500 are indicated by an asterisk.

Table 1: Gravity distance elasticity for trade and migration

	log(trade)	log(migration)			
	(1)	(2)	(3)	(4)	(5)
	1997	1850	1900	1950	2000
log(distance)	-1.353***	-2.157***	-1.798***	-1.890***	-1.511***
	(0.062)	(0.190)	(0.113)	(0.084)	(0.069)
R-squared	0.901	0.689	0.793	0.857	0.908
Observations	2,091	626	1,991	2,152	2,304

Notes: OLS estimates of equation (25) in column 1 and equation (26) in columns 2-5. Each observation is an (origin state)×(destination state)×(year) triplet for the set of 48 coterminous U.S. states. The dependent variable in column 1 is the log value of goods traded, and that in columns 2-5 is the log number of 25-74 year olds residing in the destination state in that year that were born in the origin state (as a proxy for lifetime adult migration). All specifications control for origin-year and destination-year fixed effects. Trade data comes from the Commodity Flow Survey (1997); migration data comes from the decennial U.S. census (in the years indicated). Distance is the geodesic distance between the midpoint of each state, where own distance is the expected distance between any two residents within a state, assuming the state is a square and population is uniformly distributed within the state. Standard errors are two-way clustered at the origin state and destination state. Stars indicate statistical significance (with *** denoting $p<.01$).

Table 2: Estimated productivity spillovers

	(1)	(2)	(3)
Estimated parameters:			
Contemporaneous productivity spillover (α_1)	0.409	0.217***	0.114***
	(0.963)	(0.038)	(0.034)
Historical productivity spillover (α_2)	0.054	0.045	0.040
	(0.066)	(0.033)	(0.030)
Armington elasticity of substitution (σ)	3.065		
	(4.986)		
Assumed parameter values:			
Armington elasticity of substitution (σ)	N/A	5	9
Min. Sanderson-Windmeijer first-stage F-statistic	28.5	75.9	75.9
Observations	15,640	15,640	15,640

Notes: This table reports 2SLS estimates of the parameters in equation (31) in column 1 (and subject to the stated assumed parameter values in columns 2 and 3). Each observation is a sub-county from 1850-2000. The instruments used are the interaction of linear time trends with the maximum temperature in the warmest month (and its square), and with the minimum temperature in the coldest month (and its square). All specifications control for sub-county and region-year fixed effects (where a region is one of 14 equally sized squares covering the continental U.S.) and for the excluded instruments used to estimate productivity spillovers in Table 3. Standard errors are two-way clustered at the sub-county (to allow for serial correlation over time) and county-year levels (to allow for data aggregation across sub-counties within year) and are reported in parentheses. Sanderson and Windmeijer (2016) F-statistic refers to the first-stage F-statistic for each first-stage (shown in Table C.1) obtained while partialling out the other endogenous variable; the value reported is the minimum of the two such F-statistics across the two equations. Stars indicate statistical significance (with *** denoting p<.01).

Table 3: Estimated amenity spillovers

	(1)	(2)	(3)	(4)	(5)
Estimated parameters:					
Contemporaneous amenity spillover (β_1)	1.216	0.088	-0.150	0.093	-0.145
	(1.850)	(0.284)	(0.278)	(0.284)	(0.279)
Historical amenity spillover (β_2)	0.399	0.317*	0.322*	0.325*	0.330*
	(0.335)	(0.182)	(0.178)	(0.182)	(0.179)
Migration elasticity (θ)	0.620				
	(0.858)				
Armington elasticity of substitution (σ)	0.268				
	(2.524)				
Assumed parameter values:					
Armington elasticity of substitution (σ)		5	5	9	9
Migration elasticity θ		2	4	2	4
Min. S-W first-stage F-statistic	3.078	7.373	7.373	7.373	7.373
Observations	15,640	15,640	15,640	15,640	15,640

Notes: This table reports 2SLS estimates of the parameters in equation (32) in column 1 (and subject to the stated assumed parameter values in columns 2-5). The instruments used are the interaction of linear time trends with the difference between high-intensity and low-intensity agro-climatic potential yields of maize (mean and standard deviation), and with the difference between high-intensity yields of soy and the low-intensity yields of wheat (mean and standard deviation). All specifications control for sub-county and region-year fixed effects and for the excluded instruments used to estimate productivity spillovers in Table 2. Standard errors, two-way clustered at the sub-county and county-year levels, are reported in parentheses. Notes to Table 2 contain further details. Stars indicate statistical significance (with * denoting p<.1).

Table 4: Steady-state welfare bounds

	Constants (1)	Migration costs (2)	Trade costs (3)	Local geography (4)	Scale economies (5)	Total (6)
Upper bound $\left(\ln \bar{\Omega}\right)$	1.62	0.19	0.09	0.90	1.18	3.98
Lower bound $(\ln \underline{\Omega})$	-1.59	-1.58	-0.68	-0.88	-1.33	-6.05
Difference $\left(\ln \hat{\Omega}^{PD}\right)$	3.21	1.77	0.77	1.78	2.51	10.03

Notes: This table reports the steady-state welfare bounds (and their decomposition into five exogenous components) as stated in Proposition 4. The upper and lower bound equations are:

$$\frac{1}{2}\ln c_1 + \frac{1}{2}\ln \bar{\lambda}_M^{\frac{1}{\theta}} + \frac{1}{2}\ln \bar{\lambda}_T^{\frac{1}{\sigma-1}} + \frac{1}{2}\max_i \ln \bar{A}_i \bar{u}_i + \frac{1}{2}\ln\left(\bar{L}^{\rho-\frac{1}{\theta}} N^{\varepsilon_1}\right) = \ln \bar{\Omega}$$
$$-\frac{1}{2}\ln c_2 + \frac{1}{2}\ln \underline{\lambda}_M^{\frac{1}{\theta}} + \frac{1}{2}\ln \underline{\lambda}_T^{\frac{1}{\sigma-1}} + \frac{1}{2}\min_i \ln \bar{A}_i \bar{u}_i + \frac{1}{2}\ln\left(\bar{L}^{\rho} N^{\varepsilon_2}\right) = \ln \underline{\Omega},$$

where each terms refers to the impact of the constants (column 1), the migration costs (2), the trade costs (3), the local geography (4), and the scale economies (5), respectively. The third row reports the difference between the upper bound and lower bound, i.e. $\ln \hat{\Omega}^{PD} \equiv \ln \bar{\Omega} - \ln \underline{\Omega}$, for each component and the total. Column 6 is equal to the sum of columns 1-5.

A Online Appendix: Proofs

Proofs of Propositions 1, 3, and 5 rely on Theorem 3 (parts (i) and (ii)) of Allen, Arkolakis, and Li, 2020, which we restate here for convenience:

Consider the following system of $N \times K$ system of equations

$$\prod_{h=1}^{K} \left(x_i^h\right)^{\beta_{kh}} = \sum_{j=1}^{K} K_{ij}^k \left[\prod_{h=1}^{H} \left(x_j^h\right)^{\gamma_{kh}}\right],$$

where $\{\beta_{kh}, \gamma_{kh}\}$ are known elasticities and $\left\{K_{ij}^k > 0\right\}$ are known bilateral frictions. Let $\mathbf{B} \equiv [\beta_{kh}]$ and $\mathbf{\Gamma} \equiv [\gamma_{kh}]$ be the $K \times K$ matrices of the known elasticities. Define $\mathbf{A} \equiv \mathbf{\Gamma}\mathbf{B}^{-1}$ and the absolute value (element by element) of $\mathbf{A}$ as $\mathbf{A}^p$. Then there exists a strictly positive set of $\left\{x_i^h > 0\right\}_{i\in\{1,\ldots,N\}}^{h\in\{1,\ldots,K\}}$ and that solution is unique if the spectral radius (i.e. the absolute value of the largest eigenvalue, denoted $\rho(\cdot)$) of $\mathbf{A}^p$ is weakly less than one, i.e. $\rho(\mathbf{A}^p) \leq 1$.

A.1 Proof of Proposition 1

When trade costs are symmetric, Allen and Arkolakis (2014) show that the origin and destination fixed effects of the gravity trade equation are equal up to scale. That is if $X_{ij} = K_{ij}\gamma_i\delta_j$, $K_{ij} = K_{ji}$, and $\sum_j X_{ij} = \sum_j X_{ji}$, there exists a $\kappa > 0$ such that:

$$\gamma_i = \kappa\delta_i.$$

[41]From equation (4), this implies:

$$\begin{aligned}
w_i^{1-\sigma} A_i^{\sigma-1} &= \kappa P_i^{\sigma-1} w_i L_i \iff \\
w_i^{1-\sigma} A_i^{\sigma-1} &= \kappa \left(\frac{w_i u_i}{W_i}\right)^{\sigma-1} w_i L_i \iff \\
w_i &= \kappa^{\frac{1}{1-2\sigma}} W_i^{\tilde{\sigma}} u_i^{-\tilde{\sigma}} A_i^{\tilde{\sigma}} L_i^{\frac{1}{1-2\sigma}} \iff \\
w_i &= \kappa^{\frac{1}{1-2\sigma}} W_i^{\tilde{\sigma}} \bar{u}_i^{-\tilde{\sigma}} \bar{A}_i^{\tilde{\sigma}} L_i^{\left(\alpha_1-\beta_1+\frac{1}{1-\sigma}\right)\tilde{\sigma}} \left(L_i^{lag}\right)^{(\alpha_2-\beta_2)\tilde{\sigma}}
\end{aligned}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$, and we have used the spillover functions with notation $A_i = \bar{A}_i L_i^{\alpha_1}\left(L_i^{lag}\right)^{\alpha_2}$ and $u_i = \bar{u}_i L_i^{\beta_1}\left(L_i^{lag}\right)^{\beta_2}$.

We can use this to simplify our equilibrium equations:

$$\left(W_i^{\tilde{\sigma}} u_i^{-\tilde{\sigma}} A_i^{\tilde{\sigma}} L_i^{\frac{\tilde{\sigma}}{1-\sigma}}\right)^{\sigma} L_i = \sum_j \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} W_j^{1-\sigma} \left(W_j^{\tilde{\sigma}} u_j^{-\tilde{\sigma}} A_j^{\tilde{\sigma}} L_j^{\frac{\tilde{\sigma}}{1-\sigma}}\right)^{\sigma} L_j$$

$$\Pi_i^{\theta} = \sum_j \mu_{ij}^{-\theta} W_j^{\theta}$$

$$L_i = \sum_j \mu_{ji}^{-\theta} W_i^{\theta} \Pi_j^{-\theta} L_j^{lag},$$

as the equivalent set:

[41]The exact scale (κ) is determined by the aggregate labor market clearing condition. However, the scale can be ignored by first solving for the "scaled" labor (i.e. imposing the scalar is equal to one) and then recovering the scale by imposing the labor market clearing condition. This does not affect any of the other equilibrium equations below, as they are all homogeneous of degree 0 with respect to labor.

$$
\begin{aligned}
W_i^{\tilde{\sigma}\sigma} L_i^{\tilde{\sigma}(1-\alpha_1(\sigma-1)-\beta_1\sigma)} &= \sum_j \tau_{ij}^{1-\sigma} \bar{A}_i^{(\sigma-1)\tilde{\sigma}} \bar{u}_i^{\tilde{\sigma}} \beta_j^{(\sigma-1)\tilde{\sigma}} \bar{A}_j^{\tilde{\sigma}\sigma} \left(L_i^{lag}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \\
&\quad \times \left(L_j^{lag}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} W_j^{-(\sigma-1)\tilde{\sigma}} L_j^{\tilde{\sigma}(1+\alpha_1\sigma+\beta_1(\sigma-1))}
\end{aligned}
$$

$$
\Pi_i^{\theta} = \sum_j \mu_{ij}^{-\theta} W_j^{\theta}
$$

$$
L_i W_i^{-\theta} = \sum_j \mu_{ji}^{-\theta} \Pi_j^{-\theta} L_j^{lag}.
$$

If we order the endogenous variables as L, W, Π, then the matrix of LHS coefficients in this system becomes:

$$
\boldsymbol{B} \equiv \begin{pmatrix} \tilde{\sigma}\left(1-\alpha_1\left(\sigma-1\right)-\beta_1\sigma\right) & \tilde{\sigma}\sigma & 0 \\ 0 & 0 & \theta \\ 1 & -\theta & 0 \end{pmatrix}
$$

and the equivalent for the RHS coefficients is:

$$
\boldsymbol{\Gamma} \equiv \begin{pmatrix} \tilde{\sigma}\left(1+\alpha_1\sigma+\beta_1\left(\sigma-1\right)\right) & -\left(\sigma-1\right)\tilde{\sigma} & 0 \\ 0 & \theta & 0 \\ 0 & 0 & -\theta \end{pmatrix}.
$$

Hence, we have:

$$
\mathbf{A} \equiv \boldsymbol{\Gamma}\mathbf{B}^{-1} = \begin{pmatrix} \frac{\theta-\sigma-\beta_1\theta+\alpha_1\sigma\theta+\beta_1\sigma\theta+1}{\sigma+\theta+\alpha_1\theta-\alpha_1\sigma\theta-\beta_1\sigma\theta} & 0 & \frac{\tilde{\sigma}(2\sigma-1)(\alpha_1+1)}{\sigma+\theta+\alpha_1\theta-\alpha_1\sigma\theta-\beta_1\sigma\theta} \\ \frac{\theta/\tilde{\sigma}}{\sigma+\theta+\alpha_1\theta-\alpha_1\sigma\theta-\beta_1\sigma\theta} & 0 & \frac{-\theta(\alpha_1-\alpha_1\sigma)-\beta_1\sigma+1}{\sigma+\theta+\alpha_1\theta-\alpha_1\sigma\theta-\beta_1\sigma\theta} \\ 0 & -1 & 0 \end{pmatrix}.
$$

It is straightforward to show that the spectral radius of the (element-wise) absolute value of $\mathbf{A}$, denoted $\mathbf{A}^p$, is the same as the spectral radius of the 2×2 matrix that removes the third row and second column from $\mathbf{A}^p$.[42] Hence the uniqueness condition requires that the absolute value of this smaller matrix has a spectral radius no greater than one, i.e.:

$$
\rho\begin{pmatrix} \left|\frac{\theta(1+\alpha_1\sigma+\beta_1(\sigma-1))-(\sigma-1)}{\sigma+\theta(1+(1-\sigma)\alpha_1-\beta_1\sigma)}\right| & \left|\frac{(\sigma-1)(\alpha_1+1)}{\sigma+\theta(1+(1-\sigma)\alpha_1-\beta_1\sigma)}\right| \\ \left|\frac{\theta/\tilde{\sigma}}{\sigma+\theta(1+(1-\sigma)\alpha_1-\beta_1\sigma)}\right| & \left|\frac{\theta(1-(\sigma-1)\alpha_1-\beta_1\sigma)}{\sigma+\theta(1+(1-\sigma)\alpha_1-\beta_1\sigma)}\right| \end{pmatrix} \leq 1,
$$

as required.

A.2 Proof of Proposition 2

Recall that the equilibrium of the dynamic model corresponds to the set of endogenous variables $\{L_{it}, W_{it}, \Pi_{it}\}$ that solve the following system of equations given exogenous parameters $\left\{\bar{A}_{it}, \bar{u}_{it}, \tau_{ij}, \mu_{ij}, L_{it-1}\right\}$.

[42] This result is achieved by construction. Let $\mathbf{A}^p$ denote the element-wise absolute value of $\mathbf{A}$ and let $\mathbf{A}^p_{-2,3}$ denote the 2×2 matrix that removes the third row and second column from $\mathbf{A}^p$. By the Perron Frobenius theorem, we have $\lambda\mathbf{A}^p_{-2,3} = \lambda\mathbf{x}$, where $\mathbf{x}$ is the unique (to-scale) strictly positive eigenvector corresponding to the largest eigenvalue (i.e. the spectral radius) of $\mathbf{A}^p_{-2,3}$, λ. It is straightforward to show that $\lambda\mathbf{A}^p = \lambda\tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}} \equiv \left[x_1, x_2, \frac{x_2}{\lambda}\right]$,i.e. λ is also an eigenvalue for $\mathbf{A}^p$. Moreover, because $\tilde{\mathbf{x}}$ is strictly positive—and the Perron Frobenius theorem tells us that there is only one strictly positive eigenvector (to-scale) which corresponds to the largest eigenvalue—we know that λ must be the largest eigenvalue for $\mathbf{A}^p$, i.e. λ is the spectral radius of $\mathbf{A}^p$ and $\mathbf{A}^p_{-2,3}$. We thank Vincent Lohmann for a helpful discussion about this point.

We have the (combined) trade equation:

$$L_{it}^{\tilde{\sigma}(1-\alpha_1(\sigma-1)-\beta_1)}W_{it}^{\tilde{\sigma}\sigma}=\sum_j F_{ij}L_{jt}^{\tilde{\sigma}(1+(\sigma-1)\beta+\alpha\sigma)}W_{jt}^{(1-\sigma)\tilde{\sigma}}\left(L_{i,t-1}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)}\left(L_{j,t-1}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))},\tag{A.1}$$

the population law of motion:

$$L_{it}W_{it}^{-\theta}=\sum_j \mu_{ji}^{-\theta}\Pi_{jt}^{-\theta}L_{jt-1},\tag{A.2}$$

and the value of living in a particular location (multilateral migration resistance):

$$\Pi_{it}^{\theta}=\sum_j \mu_{ij}^{-\theta}W_{jt}^{\theta}.\tag{A.3}$$

We take as given the population at time $t=0$, i.e. $\{L_{i0}\}$. The proof of Proposition 2 proceeds in five steps.

Step #1: Redefine the system

We begin by redefining the left hand side of the equilibrium equations:

$$\begin{aligned}x_{it}&\equiv L_{it}^{\tilde{\sigma}(1-\alpha(\sigma-1)-\beta_1)}W_{it}^{\tilde{\sigma}\sigma}\\ y_{it}&\equiv L_{it}W_{it}^{-\theta}\\ z_{it}&\equiv \Pi_{it}^{\theta},\end{aligned}$$

or equivalently:

$$\begin{pmatrix}\ln x_{it}\\ \ln y_{it}\\ \ln z_{it}\end{pmatrix}=\underbrace{\begin{pmatrix}\tilde{\sigma}\left(1-\alpha\left(\sigma-1\right)-\beta_1\right) & \tilde{\sigma}\sigma & 0\\ 1 & -\theta & 0\\ 0 & 0 & \theta\end{pmatrix}}_{\equiv\mathbf{B}}\begin{pmatrix}\ln L_{it}\\ \ln W_{it}\\ \ln \Pi_{it}\end{pmatrix}\iff$$

$$\mathbf{B}^{-1}\begin{pmatrix}\ln x_{it}\\ \ln y_{it}\\ \ln z_{it}\end{pmatrix}=\begin{pmatrix}\ln L_{it}\\ \ln W_{it}\\ \ln \Pi_{it}\end{pmatrix}.$$

With a slight abuse of notation, let B_{kl}^{-1} denote the $\langle k,l\rangle^{th}$ component of $\mathbf{B}^{-1}$. We can then re-write the system of equations as:

$$\begin{aligned}x_{it}=&\sum_j F_{ij}\left(x_{jt}^{B_{11}^{-1}}y_{jt}^{B_{12}^{-1}}z_{jt}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(1+(\sigma-1)\beta_1+\alpha\sigma)}\left(x_{jt}^{B_{21}^{-1}}y_{jt}^{B_{22}^{-1}}z_{jt}^{B_{23}^{-1}}\right)^{(1-\sigma)\tilde{\sigma}}\left(x_{jt-1}^{B_{11}^{-1}}y_{jt-1}^{B_{12}^{-1}}z_{jt-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))}\\ &\times\left(x_{it-1}^{B_{11}^{-1}}y_{it-1}^{B_{12}^{-1}}z_{it-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)}\\ y_{it}=&\sum_j \mu_{ji}^{-\theta}\left(x_{jt}^{B_{31}^{-1}}y_{jt}^{B_{32}^{-1}}z_{jt}^{B_{33}^{-1}}\right)^{-\theta}\left(x_{jt-1}^{B_{11}^{-1}}y_{jt-1}^{B_{12}^{-1}}z_{jt-1}^{B_{13}^{-1}}\right)^{1}\\ z_{it}=&\sum_j \mu_{ij}^{-\theta}\left(x_{jt}^{B_{21}^{-1}}y_{jt}^{B_{22}^{-1}}z_{jt}^{B_{23}^{-1}}\right)^{\theta},\end{aligned}$$

or, equivalently:

$$x_{it}=\sum_j F_{ij} x_{jt}^{\tilde{A}_{11}} y_{jt}^{\tilde{A}_{12}} z_{jt}^{\tilde{A}_{13}} \left(x_{jt-1}^{B_{11}^{-1}} y_{jt-1}^{B_{12}^{-1}} z_{jt-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \left(x_{it-1}^{B_{11}^{-1}} y_{it-1}^{B_{12}^{-1}} z_{it-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \tag{A.4}$$

$$y_{it}=\sum_j \mu_{ji}^{-\theta} x_{jt}^{\tilde{A}_{21}} y_{jt}^{\tilde{A}_{22}} z_{jt}^{\tilde{A}_{23}} x_{jt-1}^{B_{11}^{-1}} y_{jt-1}^{B_{12}^{-1}} z_{jt-1}^{B_{13}^{-1}} \tag{A.5}$$

$$z_{it}=\sum_j \mu_{ij}^{-\theta} x_{jt}^{\tilde{A}_{31}} y_{jt}^{\tilde{A}_{32}} z_{jt}^{\tilde{A}_{33}}, \tag{A.6}$$

where:

$$\tilde{\mathbf{A}}=\begin{pmatrix}\tilde{\sigma}\left(1+(\sigma-1)\beta_1+\alpha_1\sigma\right) & (1-\sigma)\tilde{\sigma} & 0\\ 0 & 0 & -\theta\\ 0 & \theta & 0\end{pmatrix}\mathbf{B}^{-1}.$$

Equations (A.4)-(A.6) constitute the redefined system. As an aside, note that $\left|\tilde{\mathbf{A}}\right| \equiv \mathbf{A}(\alpha_1,\beta_1)$, i.e. $\mathbf{A}(\alpha_1,\beta_1)$ is the (element-wise) absolute value of $\tilde{\mathbf{A}}$.

Step #2: Re-write the system in terms of changes

We can further re-write equations Equations (A.4)-(A.6) as:

$$\begin{aligned}
x_{it} =& \sum_j F_{ij} x_{jt}^{\tilde{A}_{11}} y_{jt}^{\tilde{A}_{12}} z_{jt}^{\tilde{A}_{13}} \left(x_{jt-1}^{B_{11}^{-1}} y_{jt-1}^{B_{12}^{-1}} z_{jt-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \left(x_{it-1}^{B_{11}^{-1}} y_{it-1}^{B_{12}^{-1}} z_{it-1}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \iff \\
x_{it} =& \sum_j F_{ij} \left(\frac{x_{jt}}{x_{j,t-1}}\right)^{\tilde{A}_{11}} \left(\frac{y_{jt}}{y_{j,t-1}}\right)^{\tilde{A}_{12}} \left(\frac{z_{j,t}}{z_{j,t-1}}\right)^{\tilde{A}_{13}} \left(\left(\frac{x_{j,t-1}}{x_{j,t-2}}\right)^{B_{11}^{-1}} \left(\frac{y_{j,t-1}}{y_{j,t-2}}\right)^{B_{12}^{-1}} \left(\frac{z_{j,t-1}}{z_{j,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \\
& \times \left(\left(\frac{x_{i,t-1}}{x_{i,t-2}}\right)^{B_{11}^{-1}} \left(\frac{y_{i,t-1}}{y_{i,t-2}}\right)^{B_{12}^{-1}} \left(\frac{z_{i,t-1}}{z_{i,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \\
& \times x_{j,t-1}^{\tilde{A}_{11}} y_{j,t-1}^{\tilde{A}_{12}} z_{j,t-1}^{\tilde{A}_{13}} \left(x_{jt-2}^{B_{11}^{-1}} y_{jt-2}^{B_{12}^{-1}} z_{jt-2}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \\
& \times \left(x_{it-2}^{A_{11}^{-1}} y_{it-2}^{A_{12}^{-1}} z_{it-2}^{A_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)},
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
y_{it} =& \sum_j \mu_{ji}^{-\theta} x_{jt}^{\tilde{A}_{21}} y_{jt}^{\tilde{A}_{22}} z_{jt}^{\tilde{A}_{23}} x_{jt-1}^{B_{11}^{-1}} y_{jt-1}^{B_{12}^{-1}} z_{jt-1}^{B_{13}^{-1}} \iff \\
y_{it} =& \sum_j \mu_{ji}^{-\theta} \left(\frac{x_{jt}}{x_{j,t-1}}\right)^{\tilde{A}_{21}} \left(\frac{y_{j,t}}{y_{j,t-1}}\right)^{\tilde{A}_{22}} \left(\frac{z_{j,t}}{z_{j,t-1}}\right)^{\tilde{A}_{23}} \left(\frac{x_{j,t-1}}{x_{j,t-2}}\right)^{B_{11}^{-1}} \left(\frac{y_{j,t-1}}{y_{j,t-2}}\right)^{B_{12}^{-1}} \left(\frac{z_{j,t-1}}{z_{j,t-2}}\right)^{B_{13}^{-1}} \\
& \times x_{j,t-1}^{\tilde{A}_{21}} y_{j,t-1}^{\tilde{A}_{22}} z_{j,t-1}^{\tilde{A}_{23}} x_{jt-2}^{B_{11}^{-1}} y_{jt-2}^{B_{12}^{-1}} z_{jt-2}^{B_{13}^{-1}},
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
z_{it} =& \sum_j F_{ij} x_{jt}^{\tilde{A}_{31}} y_{jt}^{\tilde{A}_{32}} z_{jt}^{\tilde{A}_{33}} \iff \\
z_{it} =& \sum_j F_{ij} \left(\frac{x_{jt}}{x_{j,t-1}}\right)^{\tilde{A}_{31}} \left(\frac{y_{jt}}{y_{j,t-1}}\right)^{\tilde{A}_{32}} \left(\frac{z_{j,t}}{z_{j,t-1}}\right)^{\tilde{A}_{33}} x_{j,t-1}^{\tilde{A}_{31}} y_{j,t-1}^{\tilde{A}_{32}} z_{j,t-1}^{\tilde{A}_{33}}.
\end{aligned} \tag{A.9}$$

Equations (A.7)-(A.9) are then the redefined system in changes.

Step #3: Bound the changes

Define the following constants:

$$
\begin{aligned}
M_{x,t} &\equiv \max_j \frac{x_{j,t}}{x_{j,t-1}},\ M_{y,t} \equiv \max_j \frac{y_{j,t}}{y_{j,t-1}},\ M_{z,t} \equiv \max_j \frac{z_{j,t}}{z_{j,t-1}} \\
m_{x,t} &\equiv \min_j \frac{x_{j,t}}{x_{j,t-1}},\ m_{y,t} \equiv \min_j \frac{y_{j,t}}{y_{j,t-1}},\ m_{z,t} \equiv \min_j \frac{z_{j,t}}{z_{j,t-1}} \\
\mu_{x,t} &\equiv \frac{M_{x,t}}{m_{x,t}},\ \mu_{y,t} \equiv \frac{M_{y,t}}{m_{y,t}},\ \mu_{z,t} \equiv \frac{M_{z,t}}{m_{z,t}}.
\end{aligned}
$$

Let us bound $\{x_{it}\}$ from above first:

$$
\begin{aligned}
x_{it} = &\sum_j F_{ij} \left(\frac{x_{jt}}{x_{j,t-1}}\right)^{\tilde{A}_{11}} \left(\frac{y_{jt}}{y_{j,t-1}}\right)^{\tilde{A}_{12}} \left(\frac{z_{j,t}}{z_{j,t-1}}\right)^{\tilde{A}_{13}} \left(\left(\frac{x_{j,t-1}}{x_{j,t-2}}\right)^{B_{11}^{-1}} \left(\frac{y_{j,t-1}}{y_{j,t-2}}\right)^{B_{12}^{-1}} \left(\frac{z_{j,t-1}}{z_{j,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \\
&\times \left(\left(\frac{x_{i,t-1}}{x_{i,t-2}}\right)^{B_{11}^{-1}} \left(\frac{y_{i,t-1}}{y_{i,t-2}}\right)^{B_{12}^{-1}} \left(\frac{z_{i,t-1}}{z_{i,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \\
&\times x_{j,t-1}^{\tilde{A}_{11}} y_{j,t-1}^{\tilde{A}_{12}} z_{j,t-1}^{\tilde{A}_{13}} \left(x_{jt-2}^{B_{11}^{-1}} y_{jt-2}^{B_{12}^{-1}} z_{jt-2}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \left(x_{it-2}^{B_{11}^{-1}} y_{it-2}^{B_{12}^{-1}} z_{it-2}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \implies \\
\frac{x_{it}}{x_{i,t-1}} \leq &\frac{M_{x,t}^{\tilde{A}_{11}\mathbf{1}\{\tilde{A}_{11}\geq 0\}}}{m_{x,t}^{-\tilde{A}_{11}\mathbf{1}\{\tilde{A}_{11}<0\}}} \frac{M_{y,t}^{\tilde{A}_{12}\mathbf{1}\{\tilde{A}_{12}\geq 0\}}}{m_{y,t}^{-\tilde{A}_{12}\mathbf{1}\{\tilde{A}_{12}<0\}}} \frac{M_{z,t}^{\tilde{A}_{13}\mathbf{1}\{\tilde{A}_{13}\geq 0\}}}{m_{z,t}^{-\tilde{A}_{13}\mathbf{1}\{\tilde{A}_{13}<0\}}} \frac{M_{x,t-1}^{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{m_{x,t-1}^{-B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times \frac{M_{x,t-1}^{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{m_{x,t-1}^{-B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}} \frac{M_{y,t-1}^{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{m_{y,t-1}^{-B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times \frac{M_{y,t-1}^{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{m_{y,t-1}^{-B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}} \frac{M_{z,t-1}^{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{m_{z,t-1}^{-B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times \frac{M_{z,t-1}^{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{m_{z,t-1}^{-B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}}.
\end{aligned}
\tag{A.10}
$$

Similarly, we can bound $\{x_{i,t}\}$ from below:

$$\begin{aligned}
x_{it} =& \sum_j F_{ij}\left(\frac{x_{jt}}{x_{j,t-1}}\right)^{\tilde{A}_{11}}\left(\frac{y_{jt}}{y_{j,t-1}}\right)^{\tilde{A}_{12}}\left(\frac{z_{j,t}}{z_{j,t-1}}\right)^{\tilde{A}_{13}}\left(\left(\frac{x_{j,t-1}}{x_{j,t-2}}\right)^{B_{11}^{-1}}\left(\frac{y_{j,t-1}}{y_{j,t-2}}\right)^{B_{12}^{-1}}\left(\frac{z_{j,t-1}}{z_{j,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \\
&\times\left(\left(\frac{x_{i,t-1}}{x_{i,t-2}}\right)^{B_{11}^{-1}}\left(\frac{y_{i,t-1}}{y_{i,t-2}}\right)^{B_{12}^{-1}}\left(\frac{z_{i,t-1}}{z_{i,t-2}}\right)^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \qquad x_{j,t-1}^{\tilde{A}_{11}}y_{j,t-1}^{\tilde{A}_{12}}z_{j,t-1}^{\tilde{A}_{13}}\left(x_{jt-2}^{B_{11}^{-1}}y_{jt-2}^{B_{12}^{-1}}z_{jt-2}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))} \\
&\times\left(x_{it-2}^{B_{11}^{-1}}y_{it-2}^{B_{12}^{-1}}z_{it-2}^{B_{13}^{-1}}\right)^{\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)} \qquad \Longrightarrow \\
\frac{x_{it}}{x_{i,t-1}} \geq& \frac{m_{x,t}^{\tilde{A}_{11}\mathbf{1}\{\tilde{A}_{11}\geq 0\}}}{M_{x,t}^{-\tilde{A}_{11}\mathbf{1}\{\tilde{A}_{11}<0\}}}\frac{m_{y,t}^{\tilde{A}_{12}\mathbf{1}\{\tilde{A}_{12}\geq 0\}}}{M_{y,t}^{-\tilde{A}_{12}\mathbf{1}\{\tilde{A}_{12}<0\}}}\frac{m_{z,t}^{\tilde{A}_{13}\mathbf{1}\{\tilde{A}_{13}\geq 0\}}}{M_{z,t}^{-\tilde{A}_{13}\mathbf{1}\{\tilde{A}_{13}<0\}}}\frac{m_{x,t-1}^{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{M_{x,t-1}^{-B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times\frac{m_{x,t-1}^{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{M_{x,t-1}^{-B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}}\frac{m_{y,t-1}^{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{M_{y,t-1}^{-B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times\frac{m_{y,t-1}^{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{M_{y,t-1}^{-B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}}\frac{m_{z,t-1}^{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\geq 0\}}}{M_{z,t-1}^{-B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))<0\}}} \\
&\times\frac{m_{z,t-1}^{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\geq 0\}}}{M_{z,t-1}^{-B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)\mathbf{1}\{B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)<0\}}}. \qquad \text{(A.11)}
\end{aligned}$$

Combining equations (A.10) and (A.11) (dividing the maximum by the minimum) implies:

$$\begin{aligned}
\mu_{x,t} \leq& \mu_{x,t}^{|\tilde{A}_{11}|}\mu_{y,t}^{|\tilde{A}_{12}|}\mu_{z,t}^{|\tilde{A}_{13}|} \\
&\times \mu_{x,t-1}^{|B_{11}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))|+|B_{11}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)|}\mu_{y,t-1}^{|B_{12}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))|+|B_{12}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)|} \\
&\times \mu_{z,t-1}^{|B_{13}^{-1}\tilde{\sigma}(\alpha_2\sigma+\beta_2(\sigma-1))|+|B_{13}^{-1}\tilde{\sigma}(\alpha_2(\sigma-1)+\beta_2\sigma)|}.
\end{aligned}$$

Proceeding similarly for $\{y_{it}\}$ and $\{z_{it}\}$ yield, respectively:

$$\mu_{y,t} \leq \mu_{x,t}^{|\tilde{A}_{21}|}\mu_{y,t}^{|\tilde{A}_{22}|}\mu_{z,t}^{|\tilde{A}_{23}|}\mu_{x,t-1}^{|B_{11}^{-1}|}\mu_{y,t-1}^{|B_{12}^{-1}|}\mu_{z,t-1}^{|B_{13}^{-1}|}$$

$$\mu_{z,t} \leq \mu_{x,t}^{|\tilde{A}_{31}|}\mu_{y,t}^{|\tilde{A}_{32}|}\mu_{z,t}^{|\tilde{A}_{33}|}.$$

Step #4: Combining the bounds

Combining the three inequalities and taking logs yields:

$$\begin{aligned}
\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix} &\leq \underbrace{\begin{pmatrix}|\tilde{A}_{11}| & |\tilde{A}_{12}| & |\tilde{A}_{13}|\\ |\tilde{A}_{21}| & |\tilde{A}_{22}| & |\tilde{A}_{23}|\\ |\tilde{A}_{31}| & |\tilde{A}_{32}| & |\tilde{A}_{33}|\end{pmatrix}}_{\equiv\mathbf{A}(\alpha_1,\beta_1)}\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix} + \mathbf{C}\begin{pmatrix}\ln\mu_{x,t-1}\\ \ln\mu_{y,t-1}\\ \ln\mu_{z,t-1}\end{pmatrix} \iff \\
(\mathbf{I}-\mathbf{A}(\alpha_1,\beta_1))\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix} &\leq \mathbf{C}\begin{pmatrix}\ln\mu_{x,t-1}\\ \ln\mu_{y,t-1}\\ \ln\mu_{z,t-1}\end{pmatrix}, \qquad \text{(A.12)}
\end{aligned}$$

for all $t > 0$, where $\mathbf{C}$ is a 3x3 matrix whose first row has elements $C_{1j} = \left|B_{1j}^{-1}\tilde{\sigma}\left(\alpha_2\sigma + \beta_2\left(\sigma-1\right)\right)\right| + \left|B_{1j}^{-1}\tilde{\sigma}\left(\alpha_2\left(\sigma-1\right) + \beta_2\sigma\right)\right|$, whose second row has elements $C_{2j} = \left|B_{1j}^{-1}\right|$, and whose third row is a vector of zeroes.

Because $\rho\left(\mathbf{A}\left(\alpha_1,\beta_1\right)\right) < 1$, $\left(\mathbf{I} - \mathbf{A}\left(\alpha_1,\beta_1\right)\right)$ in an M-matrix and is invertible, which in turn implies that its inverse $\left(\mathbf{I} - \mathbf{A}\left(\alpha_1,\beta_1\right)\right)^{-1}$ is strictly positive. As a result, we can multiply both sides of equation (A.12) by $\left(\mathbf{I} - \mathbf{A}\left(\alpha_1,\beta_1\right)\right)^{-1}$ while preserving the inequality, which yields:

$$\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix} \leq \left(\mathbf{I} - \mathbf{A}\left(\alpha_1,\beta_1\right)\right)^{-1}\mathbf{C}\begin{pmatrix}\ln\mu_{x,t-1}\\ \ln\mu_{y,t-1}\\ \ln\mu_{z,t-1}\end{pmatrix}. \tag{A.13}$$

Step #5: Combining the bounds

Finally, we convert the bound (A.13) back into $\{L_{it}, W_{it}, \Pi_{it}\}$ space. To do so, recall that:

$$\mathbf{B}^{-1}\begin{pmatrix}\ln x_{it}\\ \ln y_{it}\\ \ln z_{it}\end{pmatrix} = \begin{pmatrix}\ln L_{it}\\ \ln W_{it}\\ \ln \Pi_{it}\end{pmatrix}$$

so that, for example, we have:

$$\begin{aligned}
&\mu_{L,t} \equiv \frac{\max_i L_{i,t}/L_{i,t-1}}{\min_i L_{i,t}/L_{i,t-1}} \iff\\
&\mu_{L,t} = \frac{\max_i x_{i,t}^{B_{11}^{-1}} y_{i,t}^{B_{12}^{-1}} z_{i,t}^{B_{13}^{-1}} / x_{i,t-1}^{B_{11}^{-1}} y_{i,t-1}^{B_{12}^{-1}} z_{i,t-1}^{B_{13}^{-1}}}{\min_i x_{i,t}^{B_{11}^{-1}} y_{i,t}^{B_{12}^{-1}} z_{i,t}^{B_{13}^{-1}} / x_{i,t-1}^{B_{11}^{-1}} y_{i,t-1}^{B_{12}^{-1}} z_{i,t-1}^{B_{13}^{-1}}} \implies\\
&\mu_{L,t} \leq \frac{\max_i\left(\left(x_{i,t}/x_{i,t-1}\right)^{B_{11}^{-1}}\right) \times \max_i\left(\left(y_{i,t}/y_{i,t-1}\right)^{B_{12}^{-1}}\right) \times \max_i\left(\left(z_{i,t}/z_{i,t-1}\right)^{B_{13}^{-1}}\right)}{\min_i\left(\left(x_{i,t}/x_{i,t-1}\right)^{B_{11}^{-1}}\right) \times \min_i\left(\left(y_{i,t}/y_{i,t-1}\right)^{B_{12}^{-1}}\right) \times \min_i\left(\left(z_{i,t}/z_{i,t-1}\right)^{B_{13}^{-1}}\right)} \iff\\
&\mu_{L,t} \leq \left(\frac{\max_i\left(x_{i,t}/x_{i,t-1}\right)}{\min_i\left(x_{i,t}/x_{i,t-1}\right)}\right)^{\left|B_{11}^{-1}\right|} \times \left(\frac{\max_i\left(y_{i,t}/y_{i,t-1}\right)}{\min_i\left(y_{i,t}/y_{i,t-1}\right)}\right)^{\left|B_{12}^{-1}\right|} \times \left(\frac{\max_i\left(z_{i,t}/z_{i,t-1}\right)}{\min_i\left(z_{i,t}/z_{i,t-1}\right)}\right)^{\left|B_{13}^{-1}\right|} \iff\\
&\mu_{L,t} \leq \mu_{x,t}^{\left|B_{11}^{-1}\right|}\mu_{y,t}{}^{\left|B_{12}^{-1}\right|}\mu_{z,t}^{\left|B_{13}^{-1}\right|}.
\end{aligned}$$

Proceeding similarly for $\mu_{W,t}$ and $\mu_{\Pi,t}$ yields:

$$\begin{pmatrix}\ln\mu_{L,t}\\ \ln\mu_{W,t}\\ \ln\mu_{\Pi,t}\end{pmatrix} \leq \left|\mathbf{B}^{-1}\right|\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix}. \tag{A.14}$$

An identical argument starting with the expression $\begin{pmatrix}\ln x_{it}\\ \ln y_{it}\\ \ln z_{it}\end{pmatrix} = \mathbf{B}\begin{pmatrix}\ln L_{it}\\ \ln W_{it}\\ \ln \Pi_{it}\end{pmatrix}$ yields:

$$\begin{pmatrix}\ln\mu_{x,t}\\ \ln\mu_{y,t}\\ \ln\mu_{z,t}\end{pmatrix} \leq \left|\mathbf{B}\right|\begin{pmatrix}\ln\mu_{L,t}\\ \ln\mu_{W,t}\\ \ln\mu_{\Pi,t}\end{pmatrix}. \tag{A.15}$$

Substituting bounds (A.14) and (A.15) into bound (A.13) yields:

$$\begin{pmatrix}\ln\mu_{L,t}\\ \ln\mu_{W,t}\\ \ln\mu_{\Pi,t}\end{pmatrix} \leq \left|\mathbf{B}^{-1}\right|\left(\mathbf{I} - \tilde{\mathbf{A}}\left(\alpha_1,\beta_1\right)\right)^{-1}\mathbf{C}\left|\mathbf{B}\right|\begin{pmatrix}\ln\mu_{L,t-1}\\ \ln\mu_{W,t-1}\\ \ln\mu_{\Pi,t-1}\end{pmatrix},$$

as required.

A.3 Proof of Proposition 3

The proof of the first part of the proposition (sufficient conditions for uniqueness) proceeds similarly to the proof of Proposition 1. If migration costs are symmetric and we are in the steady-state, we have: $\sum_i L_{ij} = \sum_j L_{ji}$, $L_{ij} = M_{ij} g_i d_j$, and $M_{ij} = M_{ji}$. So then it will be the case that:

$$g_i \propto d_i.$$

In our case, this implies:

$$W_i \Pi_i L_i^{-\frac{1}{\theta}} = \Omega^2,$$

which recall is our measure of steady-state welfare.

This simplifies our system of equations as follows:

$$W_i^{\tilde{\sigma}\sigma} L_i^{\tilde{\sigma}(1-(\alpha_1+\alpha_2)(\sigma-1)-\sigma(\beta_1+\beta_2))} = \sum_j \tau_{ij}^{1-\sigma} \bar{A}_i^{(\sigma-1)\tilde{\sigma}} \bar{u}_i^{\tilde{\sigma}} u_j^{(\sigma-1)\tilde{\sigma}} \bar{A}_j^{\tilde{\sigma}\sigma} W_j^{-(\sigma-1)\tilde{\sigma}} L_j^{\tilde{\sigma}(1+(\alpha_1+\alpha_2)\sigma+(\beta_1+\beta_2)(\sigma-1))}$$

$$L_i W_i^{-\theta} = \left(\Omega^2\right)^{-\theta} \sum_j \mu_{ij}^{-\theta} W_j^{\theta}.$$

Let us order the endogenous variables as L, W. Define $\tilde{\alpha} \equiv \alpha_1 + \alpha_2$ and $\tilde{\beta} \equiv \beta_1 + \beta_2$ Then the matrix of LHS coefficients becomes:

$$\boldsymbol{B} \equiv \begin{pmatrix} \tilde{\sigma}\left(1 - \tilde{\alpha}\left(\sigma-1\right) - \tilde{\beta}\sigma\right) & \tilde{\sigma}\sigma \\ 1 & -\theta \end{pmatrix},$$

and the matrix on the RHS coefficients becomes:

$$\boldsymbol{\Gamma} \equiv \begin{pmatrix} \tilde{\sigma}\left(1 + \tilde{\alpha}\sigma + \tilde{\beta}\left(\sigma-1\right)\right) & -\left(\sigma-1\right)\tilde{\sigma} \\ 0 & \theta \end{pmatrix}.$$

Hence, we have:

$$\mathbf{A} \equiv \mathbf{\Gamma}\mathbf{B}^{-1} = \begin{pmatrix} \frac{\theta-\sigma-\tilde{\beta}\theta+\tilde{\alpha}\sigma\theta+1}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)} & \frac{-(\sigma-1)(\tilde{\alpha}+1)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)} \\ \frac{\theta/\tilde{\sigma}}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)} & \frac{-\theta\left(\tilde{\alpha}(1-\sigma)-\tilde{\beta}\sigma+1\right)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)} \end{pmatrix}.$$

As a result, the condition for uniqueness is identical to that above, where we simply replace α_1 and β_1 with $\tilde{\alpha} \equiv (\alpha_1 + \alpha_2)$ and $\tilde{\beta} \equiv (\beta_1 + \beta_2)$, as required:

$$\rho\begin{pmatrix} \left|\frac{\theta\left(1+\tilde{\alpha}\sigma+\tilde{\beta}(\sigma-1)\right)-(\sigma-1)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| & \left|\frac{(\sigma-1)(\tilde{\alpha}+1)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| \\ \left|\frac{\theta/\tilde{\sigma}}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| & \left|\frac{\theta\left(1-(\sigma-1)\tilde{\alpha}-\tilde{\beta}\sigma\right)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| \end{pmatrix} \leq 1.$$

The second part of the proposition claims that there exists a geography for which if

$$\rho\begin{pmatrix} \left|\frac{\theta\left(1+\tilde{\alpha}\sigma+\tilde{\beta}(\sigma-1)\right)-(\sigma-1)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| & \left|\frac{(\sigma-1)(\tilde{\alpha}+1)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| \\ \left|\frac{\theta/\tilde{\sigma}}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| & \left|\frac{\theta\left(1-(\sigma-1)\tilde{\alpha}-\tilde{\beta}\sigma\right)}{\sigma+\theta\left(1+(1-\sigma)\tilde{\alpha}-\tilde{\beta}\sigma\right)}\right| \end{pmatrix} > 1,$$

then there exist multiple equilibria. For readability, we present it this result as a general theorem, under which our model clearly falls:

Theorem 1. *Consider the following mathematical system:*

$$x_{i,1} = \lambda_1 \sum_{j=1}^{N} K_{ij,1} x_{j,1}^{a_{11}} x_{j,2}^{a_{12}} \tag{A.16}$$

$$x_{i,2} = \lambda_2 \sum_{j=1}^{N} K_{ij,2} x_{j,1}^{a_{21}} x_{j,2}^{a_{22}}, \tag{A.17}$$

where $\{K_{ij,k}\}_{i,j\in\{1,\dots,N\}}^{l\in\{1,2\}}$ *are the "kernels" of (exogenous) bilateral frictions,* $\{a_{lk}\}_{l,k\in\{1,2\}}$ *are (exogenous) elasticities,* $\{x_{i,k}\}_{i\in\{1,\dots,N\}}^{k\in\{1,2\}}$ *are (endogenous) strictly positive vectors and* $\{\lambda_k\}_{k\in\{1,2\}}$ *are either endogenous scalars determined by additional constraints or are exogenous. If the spectral radius of the* 2×2 *matrix* $\mathbf{A}^p \equiv [|a_{kl}|]$ *is greater than one, then there exist kernels* $\{K_{ij,k}\}_{i,j\in\{1,\dots,N\}}^{k\in\{1,2\}}$ *such that there are multiple solutions to equations (A.16) and (A.17).*

Proof. The proof proceeds by construction. We begin by performing two transformations of the problem that simplifies the setup. First, we absorb the scalars into the endogenous variables. To do so, define $y_{i,k} = \left(\lambda_1^{d_{k,1}} \lambda_2^{d_{k,2}}\right) x_{i,k}$, where $\mathbf{D} = [d_{kl}] \equiv -(\mathbf{I} - \mathbf{A})^{-1}$. Note that this is well defined as long as the spectral radius of $\mathbf{A}$ is not equal to one. It is straightforward to then show that the following equations:

$$y_{i,1} = \sum_j K_{ij,1} y_{j,1}^{a_{11}} y_{j,2}^{a_{12}}$$

$$y_{i,2} = \sum_j K_{ij,2} y_{j,1}^{a_{21}} y_{j,2}^{a_{22}}$$

are equivalent to equations (A.16) and (A.17). To see this, substitute in the definition of $y_{i,k}$, yielding:

$$\left(\lambda_1^{d_{11}} \lambda_2^{d_{12}}\right) x_{i,1} = \sum_j K_{ij,1} x_{j,1}^{a_{11}} \left(\lambda_1^{d_{11}} \lambda_2^{d_{12}}\right)^{a_{11}} x_{j,2}^{a_{12}} \left(\lambda_1^{d_{21}} \lambda_2^{d_{22}}\right)^{a_{12}}$$

$$\left(\lambda_1^{d_{21}} \lambda_2^{d_{22}}\right) x_{i,2} = \sum_j K_{ij,2} y_{j,1}^{a_{21}} \left(\lambda_1^{d_{11}} \lambda_2^{d_{12}}\right)^{a_{21}} y_{j,2}^{a_{22}} \left(\lambda_1^{d_{21}} \lambda_2^{d_{22}}\right)^{a_{22}},$$

which, rearranging, yields:

$$x_{i,1} = \lambda_1^{-d_{11}+a_{11}d_{11}+a_{12}d_{21}} \lambda_2^{-d_{12}+a_{11}d_{12}+a_{12}d_{22}} \sum_j K_{ij,1} x_{j,1}^{a_{11}} x_{j,2}^{a_{12}}$$

$$x_{i,2} = \lambda_1^{-d_{21}+a_{21}d_{11}+a_{22}d_{21}} \lambda_2^{-d_{22}+a_{21}d_{12}+a_{22}d_{22}} \sum_j K_{ij,2} y_{j,1}^{a_{21}} y_{j,2}^{a_{22}},$$

which, given the definition of $\mathbf{D}$, is equivalent to equations (A.16) and (A.17) as claimed.[43]

The second transformation is closely related to the "exact hat" algebra pioneered by Dekle, Eaton, and Kortum (2008) in the field of trade and considers a "normalized" system of equations around an observed equilibrium. Suppose we observe a steady-state solution $\{y_{i,k}\}_{i\in S, k\in\{1,.2\}}$ that satisfies:

$$y_{i,1} = \sum_j K_{ij,1} y_{j,1}^{a_{11}} y_{j,2}^{a_{12}}$$

$$y_{i,2} = \sum_j K_{ij,2} y_{j,1}^{a_{21}} y_{j,2}^{a_{22}}.$$

We are interested in knowing whether there exists a different steady-state solution $\{x_{i,k}\}_{i\in S, k\in\{1,.2\}}$ that also

[43] This follows because $\exp\left(\left(-\mathbf{D} + \mathbf{AD}\right) \ln \boldsymbol{\lambda}\right) = \exp\left(\left(-\left(\mathbf{I} - \mathbf{A}\right)\mathbf{D}\right) \ln \boldsymbol{\lambda}\right) = \lambda.$

satisfies the same equations:

$$x_{i,1} = \sum_j K_{ij,1} x_{j,1}^{a_{11}} x_{j,2}^{a_{12}}$$
$$x_{i,2} = \sum_j K_{ij,2} x_{j,1}^{a_{21}} x_{j,2}^{a_{22}}$$

Define $z_{i,k} \equiv \frac{x_{i,k}}{y_{i,k}}$ and note that the previous equations can be written as:

$$z_{i,1} = \sum_j F_{ij,1} z_{j,1}^{a_{11}} z_{j,2}^{a_{12}} \tag{A.18}$$
$$z_{i,2} = \sum_j F_{ij,2} z_{j,1}^{a_{21}} z_{j,2}^{a_{22}}, \tag{A.19}$$

where $F_{ij,k} \equiv \left(\frac{K_{ij,k}}{y_{i,k}} y_{j,1}^{a_{k1}} y_{j,2}^{a_{k2}}\right)$. By construction, note that $z_{i,k} = 1$ is a solution to this system of equations. Moreover, the matrices $\mathbf{F}_k$ are stochastic, i.e.:

$$\sum_j F_{ij,k} = 1 \ \forall i \in \{1, ..., N\} \, k \in \{1, 2\}.$$

In what follows, we will search for stochastic matrices $\mathbf{F}_k$ that have two solutions: one in which $z_{i,k} = 1$ for all $i \in \{1, .., N\}$ and $k \in \{1, 2\}$ and another in which there exists a $z_{i,k} \neq 1$.

It turns out to do this requires $N = 4$. Choose any $m_k < 1 < M_k$ for $k \in \{1, 2\}$. Then we will construct a set of kernels that have the following solution:

$$\begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \\ z_{3,1} & z_{3,2} \\ z_{4,1} & z_{4,2} \end{pmatrix} = \begin{pmatrix} \tilde{m}_1^A & \tilde{m}_2^A \\ \tilde{m}_1^B & \tilde{m}_2^B \\ \tilde{m}_1^C & \tilde{m}_2^C \\ \tilde{m}_1^D & \tilde{m}_2^D \end{pmatrix} = \begin{pmatrix} m_1^{\mathbf{1}\{a_{11}>0\}} M_1^{\mathbf{1}\{a_{11}\leq 0\}}; & m_2^{\mathbf{1}\{a_{12}>0\}} M_2^{\mathbf{1}\{a_{12}\leq 0\}} \\ m_1^{\mathbf{1}\{a_{21}>0\}} M_1^{\mathbf{1}\{a_{21}\leq 0\}}; & m_2^{\mathbf{1}\{a_{22}>0\}} M_2^{\mathbf{1}\{a_{22}\leq 0\}} \\ m_1^{\mathbf{1}\{a_{11}\leq 0\}} M_1^{\mathbf{1}\{a_{11}>0\}}; & m_2^{\mathbf{1}\{a_{12}\leq 0\}} M_2^{\mathbf{1}\{a_{12}>0\}} \\ m_1^{\mathbf{1}\{a_{21}\leq 0\}} M_1^{\mathbf{1}\{a_{21}>0\}}; & m_2^{\mathbf{1}\{a_{22}\leq 0\}} M_2^{\mathbf{1}\{a_{22}>0\}} \end{pmatrix}. \tag{A.20}$$

Before constructing the kernel, we note the following helpful properties.

First, define $\ln \mathbf{m} \equiv \begin{pmatrix} \ln m_1 \\ \ln m_2 \end{pmatrix}$, $\ln \mathbf{M} \equiv \begin{pmatrix} \ln M_1 \\ \ln M_2 \end{pmatrix}$, and the indicator matrix

$$\mathbf{P} \equiv \begin{pmatrix} \mathbf{1}\{a_{11} > 0\} & \mathbf{1}\{a_{12} > 0\} \\ \mathbf{1}\{a_{21} > 0\} & \mathbf{1}\{a_{22} > 0\} \end{pmatrix}$$

(for "positive"); and $\mathbf{E} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then note that we can bound $\mathbf{m}$ and $\mathbf{M}$ as follows:

$$\begin{gathered}
(\mathbf{A} \circ \mathbf{P}) \ln \mathbf{m} + (\mathbf{A} \circ (\mathbf{E}{-}\mathbf{P})) \ln \mathbf{M} \leq \ln \mathbf{m} \leq \ln \mathbf{M} \leq (\mathbf{A} \circ (\mathbf{1}{-}\mathbf{P})) \ln \mathbf{m} + (\mathbf{A} \circ \mathbf{P}) \ln \mathbf{M} \iff \\
(\mathbf{A} \circ \mathbf{P}) \ln \mathbf{m} + (\mathbf{A} - (\mathbf{A}{\circ}\mathbf{P})) \ln \mathbf{M} \leq \ln \mathbf{m} \leq \ln \mathbf{M} \leq (\mathbf{A} - (\mathbf{A}{\circ}\mathbf{P})) \ln \mathbf{m} + (\mathbf{A} \circ \mathbf{P}) \ln \mathbf{M} \iff \\
\mathbf{A} \ln \mathbf{M} - (\mathbf{A} \circ \mathbf{P}) (\ln \mathbf{M} - \ln \mathbf{m}) \leq \ln \mathbf{m} \leq \ln \mathbf{M} \leq \mathbf{A} \ln \mathbf{m} + (\mathbf{A}{\circ}\mathbf{P}) (\ln \mathbf{M} - \ln \mathbf{m}) \iff \\
\ln \mathbf{B} - (\mathbf{A} \circ \mathbf{P}) (\ln \mathbf{M} - \ln \mathbf{m}) \leq \ln \mathbf{m} \leq \ln \mathbf{M} \leq \ln \mathbf{b} + (\mathbf{A}{\circ}\mathbf{P}) (\ln \mathbf{M} - \ln \mathbf{m}) \iff \\
\ln \mathbf{B} - \ln \mathbf{D} \leq \ln \mathbf{m} \leq \ln \mathbf{M} \leq \ln \mathbf{b} + \ln \mathbf{D},
\end{gathered} \tag{A.21}$$

where $\ln \mathbf{D} \equiv (\mathbf{A} \circ \mathbf{P}) (\ln \mathbf{M} - \ln \mathbf{m}) = \begin{pmatrix} \ln \left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{11}>0\}} \left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{12}>0\}} \\ \ln \left(\frac{M_1}{m_1}\right)^{a_{21}\mathbf{1}\{a_{21}>0\}} \left(\frac{M_2}{m_2}\right)^{a_{22}\mathbf{1}\{a_{22}>0\}} \end{pmatrix}$ and $D_k \equiv \exp\left((\ln \mathbf{D})_k\right)$.

Second, we note the existence and uniqueness of weights that can be used to relate the $\tilde{m}_k^n$ ($n \in \{A, B, C, D\}$) variables to other endogenous objects. In what follows, we define those weights for $\tilde{m}_k^A$, but the corresponding results also hold for $\tilde{m}_k^B$, $\tilde{m}_k^C$, and $\tilde{m}_k^D$. Since $m_k \leq \tilde{m}_k^A \leq M_k$, then there exists a weight $C_k^A \in [0,1]$ such that:

$$\tilde{m}_k^A = C_k^A m_k + \left(1 - C_k^A\right) M_k$$

and there exists a weight $c_k^A \in [0,1]$ such that:

$$\begin{aligned}
\ln \tilde{m}_k^A &= c_k^A \ln m_k + \left(1 - c_k^A\right) \ln M_k \iff \\
\tilde{m}_k^A &= m_k^{c_k^A} M_k^{1-c_k^A} \iff \\
\tilde{m}_k^A &= M_k \left(\frac{M_k}{m_k}\right)^{-c_k^A},
\end{aligned} \tag{A.22}$$

or conversely:

$$\tilde{m}_k^A = m_k \left(\frac{M_k}{m_k}\right)^{1-c_k^A}. \tag{A.23}$$

Note that because $\tilde{m}_k^A = M_k \left(\frac{M_k}{m_k}\right)^{-c_k^A}$ from equation (A.22) we can write:

$$\begin{aligned}
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= \left(M_1 \left(\frac{M_1}{m_1}\right)^{-c_1^A}\right)^{a_{11}} \left(M_2 \left(\frac{M_2}{m_2}\right)^{-c_2^A}\right)^{a_{12}} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= M_1^{a_{11}} M_2^{a_{12}} \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{-c_1^A} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{-c_2^A} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= \frac{B_1}{D_1} D_1 \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{-c_1^A} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{-c_2^A} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= \frac{B_1}{D_1} \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^A} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^A}.
\end{aligned} \tag{A.24}$$

Similarly, we have:

$$\left(\tilde{m}_1^A\right)^{a_{21}} \left(\tilde{m}_2^A\right)^{a_{22}} = \frac{B_2}{D_2} \left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\mathbf{1}\{a_{21}>0\}-c_1^A} \left(\left(\frac{M_2}{m_2}\right)^{a_{22}}\right)^{\mathbf{1}\{a_{22}>0\}-c_2^A}. \tag{A.25}$$

Because $\tilde{m}_k^A = m_k \left(\frac{M_k}{m_k}\right)^{1-c_k^A}$ from equation (A.23) we can write:

$$\begin{aligned}
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= \left(m_1 \left(\frac{M_1}{m_1}\right)^{\left(1-c_1^A\right)}\right)^{a_{11}} \left(m_2 \left(\frac{M_2}{m_2}\right)^{\left(1-c_2^A\right)}\right)^{a_{12}} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= m_1^{a_{11}} m_2^{a_{12}} \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\left(1-c_1^A\right)} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^A\right)} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= b_1 D_1 \frac{\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\left(1-c_1^A\right)} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^A\right)}}{D_1} \iff \\
\left(\tilde{m}_1^A\right)^{a_{11}} \left(\tilde{m}_2^A\right)^{a_{12}} &= b_1 D_1 \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\left(1-c_1^A\right)-\mathbf{1}\{a_{11}>0\}} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^A\right)-\mathbf{1}\{a_{12}>0\}},
\end{aligned} \tag{A.26}$$

and similarly:

$$\left(\tilde{m}_1^A\right)^{a_{21}} \left(\tilde{m}_2^A\right)^{a_{22}} = b_2 D_2 \left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\left(1-c_1^A\right)-\mathbf{1}\{a_{11}>0\}} \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^A\right)-\mathbf{1}\{a_{12}>0\}} \tag{A.27}$$

As a result, the system of equations (A.18) and (A.19) becomes:

$$\begin{aligned}\tilde{m}_1^A =& F_{11,1}\frac{B_1}{D_1} + F_{12,1}\frac{B_1}{D_1}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^B}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^B}\\ &+ F_{13,1}b_1D_1 + F_{14,1}b_1D_1\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{(1-c_1^D)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^D)-\mathbf{1}\{a_{12}>0\}}\end{aligned}$$

$$\begin{aligned}\tilde{m}_1^B =& F_{21,1}\frac{B_1}{D_1} + F_{22,1}\frac{B_1}{D_1}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^B}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^B}\\ &+ F_{23,1}b_1D_1 + F_{24,1}b_1D_1\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{(1-c_1^D)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^D)-\mathbf{1}\{a_{12}>0\}}\end{aligned}$$

$$\begin{aligned}\tilde{m}_1^C =& F_{31,1}\frac{B_1}{D_1} + F_{32,1}\frac{B_1}{D_1}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^B}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^B}\\ &+ F_{33,1}b_1D_1 + F_{34,1}b_1D_1\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{(1-c_1^D)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^D)-\mathbf{1}\{a_{12}>0\}}\end{aligned}$$

$$\begin{aligned}\tilde{m}_1^D =& F_{41,1}\frac{B_1}{D_1} + F_{42,1}\frac{B_1}{D_1}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^B}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^B}\\ &+ F_{43,1}b_1D_1 + F_{44,1}b_1D_1\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{(1-c_1^D)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^D)-\mathbf{1}\{a_{12}>0\}}\end{aligned}$$

$$\begin{aligned}\tilde{m}_2^A =& F_{11,2}\frac{B_2}{D_2}\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\mathbf{1}\{a_{21}>0\}-c_1^A}\left(\left(\frac{M_2}{m_2}\right)^{a_{22}}\right)^{\mathbf{1}\{a_{22}>0\}-c_2^A}\\ &+ F_{12,2}\frac{B_2}{D_2} + F_{13,2}b_2D_2\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{(1-c_1^C)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^C)-\mathbf{1}\{a_{12}>0\}} + F_{14,2}b_2D_2\end{aligned}$$

$$\begin{aligned}\tilde{m}_2^B =& F_{21,2}\frac{B_2}{D_2}\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\mathbf{1}\{a_{21}>0\}-c_1^A}\left(\left(\frac{M_2}{m_2}\right)^{a_{22}}\right)^{\mathbf{1}\{a_{22}>0\}-c_2^A}\\ &+ F_{22,2}\frac{B_2}{D_2} + F_{23,2}b_2D_2\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{(1-c_1^C)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^C)-\mathbf{1}\{a_{12}>0\}} + F_{24,2}b_2D_2\end{aligned}$$

$$\begin{aligned}\tilde{m}_2^C =& F_{31,2}\frac{B_2}{D_2}\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\mathbf{1}\{a_{21}>0\}-c_1^A}\left(\left(\frac{M_2}{m_2}\right)^{a_{22}}\right)^{\mathbf{1}\{a_{22}>0\}-c_2^A} + F_{32,2}\frac{B_2}{D_2}\\ &+ F_{33,2}b_2D_2\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{(1-c_1^C)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{(1-c_2^C)-\mathbf{1}\{a_{12}>0\}} + F_{34,2}b_2D_2\end{aligned}$$

$$\tilde{m}_2^D = F_{41,2}\frac{B_2}{D_2}\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\mathbf{1}\{a_{21}>0\}-c_1^A}\left(\left(\frac{M_2}{m_2}\right)^{a_{22}}\right)^{\mathbf{1}\{a_{22}>0\}-c_2^A}$$
$$+ F_{42,2}\frac{B_2}{D_2} + F_{43,2}b_2D_2\left(\left(\frac{M_1}{m_1}\right)^{a_{21}}\right)^{\left(1-c_1^C\right)-\mathbf{1}\{a_{11}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^C\right)-\mathbf{1}\{a_{12}>0\}} + F_{44,2}b_2D_2.$$

We now move on to constructing the kernel. Note that given the inequality (A.21), there exist constants $P_k \in (0,1)$ and $Q_k \in (0,1)$ such that:

$$m_k = P_k\frac{B_k}{D_k} + (1-P_k)\,b_kD_k \tag{A.28}$$

$$M_k = Q_k\frac{B_k}{D_k} + (1-Q_k)\,b_kD_k. \tag{A.29}$$

Combining the last two results (where again we focus on $\tilde{m}_k^A$, but the following holds for $\tilde{m}_k^B$, $\tilde{m}_k^C$, and $\tilde{m}_k^D$ as well) note that:

$$\tilde{m}_k^A = C_k^A m_k + \left(1-C_k^A\right)M_k$$

and

$$m_k = P_k\frac{B_k}{D_k} + (1-P_k)\,b_kD_k$$

$$M_k = Q_k\frac{B_k}{D_k} + (1-Q_k)\,b_kD_k$$

so that:

$$\tilde{m}_k^A = C_k^A\left(P_k\frac{B_k}{D_k} + (1-P_k)\,b_kD_k\right) + \left(1-C_k^A\right)\left(Q_k\frac{B_k}{D_k} + (1-Q_k)\,b_kD_k\right) \iff$$
$$\tilde{m}_k^A = \left(\left(C_k^AP_k + \left(1-C_k^A\right)Q_k\right)\right)\frac{B_k}{D_k} + \left(C_k^A\left(1-P_k\right) + \left(1-C_k^A\right)\left(1-Q_k\right)\right)b_kD_k.$$

Moreover, noting that:

$$\left(\left(C_k^AP_k + \left(1-C_k^A\right)Q_k\right)\right) + \left(C_k^A\left(1-P_k\right) + \left(1-C_k^A\right)\left(1-Q_k\right)\right) = 1,$$

we know that $\tilde{m}_k^A$ can also be written as weighted average of $\frac{B_k}{D_k}$ and b_kD_k, with the weight being $\omega_k^A \equiv \left(\left(C_k^AP_k + \left(1-C_k^A\right)Q_k\right)\right)$.

With all of these properties established, we have enough information to define our kernels:

$$\mathbf{F}_1 = \begin{pmatrix} \omega_1^A & 0 & 1-\omega_1^A & 0 \\ \omega_1^B & 0 & 1-\omega_1^B & 0 \\ \omega_1^C & 0 & 1-\omega_1^C & 0 \\ \omega_1^D & 0 & 1-\omega_1^D & 0 \end{pmatrix}$$

$$\mathbf{F}_2 = \begin{pmatrix} 0 & \omega_2^A & 0 & 1-\omega_2^A \\ 0 & \omega_2^B & 0 & 1-\omega_2^B \\ 0 & \omega_2^C & 0 & 1-\omega_2^C \\ 0 & \omega_2^D & 0 & 1-\omega_2^D \end{pmatrix}.$$

Note that the $z_{i,k} = 1$ for all $i \in \{1,..,4\}$ and $k \in \{1,2\}$ trivially satisfies the equilibrium system. But it is also straightforward to confirm that the proposed solution (A.20) is also an equilibrium. This is because every equation has a term of $\left(\frac{B_k}{D_k}\right)$ and (b_kD_k), which we know every endogenous variable is a weighted average of (see equations (A.28) and (A.29)).

Finally, we mention that there are many geographies that deliver this multiplicity for two reasons. First, the argument above holds for any choice of $m_k < 1 < M_k$. Second, it is straightforward to show that perturbations of the above kernel also generate multiple equilibria. Suppose we considered the perturbed system of equations:

$$\mathbf{F}_1 = \begin{pmatrix} \omega_1^A - \kappa\varepsilon & \delta\varepsilon & 1-\omega_1^A - (1-\kappa)\,\varepsilon & (1-\delta)\,\varepsilon \\ \omega_1^B & 0 & 1-\omega_1^B & 0 \\ \omega_1^C & 0 & 1-\omega_1^C & 0 \\ \omega_1^D & 0 & 1-\omega_1^D & 0 \end{pmatrix},$$

where $\varepsilon > 0$, $\kappa \in [0,1]$ and $\delta \in [0,1]$. The only restriction we place is that $\omega_1^A - \kappa\varepsilon > 0 \iff \kappa\varepsilon < \omega_1^A$ and $\left(1-\omega_1^A - (1-\kappa)\,\varepsilon\right) > 0 \iff \varepsilon\,(1-\kappa) < 1-\omega_1^A$. Note that both of these equations will hold for sufficiently small ε, as $\omega_k^A = \left(C_k^l P_k + \left(1-C_k^l\right) Q_k\right)$ and $P_k \in (0,1)$ and $Q_k \in (0,1)$. In what follows, we show for any choice of $\varepsilon > 0$ (that is sufficiently small to satisfy these inequalities) and any choice of $\delta \in [0,1]$, there exists a $\kappa \in [0,1]$ that ensures the multiplicity still holds.

Then the relevant equation becomes:

$$\begin{aligned} \tilde{m}_1^A =& \omega_1^A \frac{B_1}{D_1} - \kappa\varepsilon\left(\frac{B_1}{D_1}\right) + \delta\varepsilon\left(\frac{B_1}{D_1}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-c_1^B}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-c_2^B}\right) \\ &+ \left(1-\omega_1^A\right) b_1 D_1 - (1-\kappa)\,\varepsilon b_1 D_1 + (1-\delta)\,\varepsilon b_1 D_1 \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\left(1-c_1^D\right)-\mathbf{1}\{a_{11}>0\}} \\ &+ \left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\left(1-c_2^D\right)-\mathbf{1}\{a_{12}>0\}} \iff \\ \kappa\frac{B_1}{D_1} + (1-\kappa)\, b_1 D_1 =& \delta\left(\frac{B_1}{D_1}\right) G + (1-\delta)\frac{1}{G}\left(b_1 D_1\right) \end{aligned}$$

where $G \equiv \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-\mathbf{1}\{a_{21}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-\mathbf{1}\{a_{22}>0\}}$. Recall that

$$\frac{B_1}{D_1} = \frac{M_1^{a_{11}} M_2^{a_{12}}}{\left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{11}>0\}}\left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{12}>0\}}}$$

$$b_1 D_1 = m_1^{a_{11}} m_2^{a_{12}}\left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{11}>0\}}\left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{12}>0\}},$$

i.e. B_1/D_1 is always the lowest that can be achieved given the signs of the exponents, and $b_1 D_1$ is the highest that can be achieved given the signs of the exponents. As a result, we have:

$$\begin{aligned} G\left(\frac{B_1}{D_1}\right) &= \left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-\mathbf{1}\{a_{21}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-\mathbf{1}\{a_{22}>0\}} \times \frac{M_1^{a_{11}} M_2^{a_{12}}}{\left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{11}>0\}}\left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{12}>0\}}} \iff \\ G\left(\frac{B_1}{D_1}\right) &= \frac{M_1^{a_{11}} M_2^{a_{12}}}{\left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{21}>0\}}\left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{22}>0\}}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{G}\left(b_1 D_1\right) &= \frac{m_1^{a_{11}} m_2^{a_{12}}\left(\frac{M_1}{m_1}\right)^{a_{11}\mathbf{1}\{a_{11}>0\}}\left(\frac{M_2}{m_2}\right)^{a_{12}\mathbf{1}\{a_{12}>0\}}}{\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{11}>0\}-\mathbf{1}\{a_{21}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{12}>0\}-\mathbf{1}\{a_{22}>0\}}} \iff \\ \frac{1}{G}\left(b_1 D_1\right) &= m_1^{a_{11}} m_2^{a_{12}}\left(\left(\frac{M_1}{m_1}\right)^{a_{11}}\right)^{\mathbf{1}\{a_{21}>0\}}\left(\left(\frac{M_2}{m_2}\right)^{a_{12}}\right)^{\mathbf{1}\{a_{22}>0\}}. \end{aligned}$$

Together these statements imply that:

$$\frac{B_1}{D_1} \leq G\left(\frac{B_1}{D_1}\right), \frac{1}{G}\left(b_1 D_1\right) \leq b_1 D_1,$$

since B_1/D_1 and $b_1 D_1$ are designed to be the lowest and highest (respectively) given the signs of the exponents. As a result, there exist constants (weights) $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$ such that:

$$G\left(\frac{B_1}{D_1}\right) = \lambda_1 \frac{B_1}{D_1} + \left(1-\lambda_1\right) b_1 D_1$$

$$\frac{1}{G}\left(b_1 D_1\right) = \lambda_2 \frac{B_1}{D_1} + \left(1-\lambda_2\right) b_1 D_1.$$

We now return to the above equation:

$$\begin{aligned}
&\kappa \frac{B_1}{D_1} + (1-\kappa) b_1 D_1 = \delta\left(\frac{B_1}{D_1}\right) G + (1-\delta)\frac{1}{G}\left(b_1 D_1\right) \iff \\
&\kappa \frac{B_1}{D_1} + (1-\kappa) b_1 D_1 = \delta\left(\lambda_1 \frac{B_1}{D_1} + \left(1-\lambda_1\right) b_1 D_1\right) + (1-\delta)\left(\lambda_2 \frac{B_1}{D_1} + \left(1-\lambda_2\right) b_1 D_1\right) \iff \\
&\kappa \frac{B_1}{D_1} + (1-\kappa) b_1 D_1 = \left(\delta\lambda_1 + (1-\delta)\lambda_2\right)\frac{B_1}{D_1} + \left(\delta\left(1-\lambda_1\right) + (1-\delta)\left(1-\lambda_2\right)\right) b_1 D_1. \qquad \text{(A.30)}
\end{aligned}$$

Choose $\kappa \equiv \delta\lambda_1 + (1-\delta)\lambda_2$. Then

$$\begin{aligned}
1-\kappa &= 1 - \delta\lambda_1 - (1-\delta)\lambda_2 \iff \\
1-\kappa &= 1 + \delta - \delta - \delta\lambda_1 - (1-\delta)\lambda_2 \iff \\
1-\kappa &= \delta\left(1-\lambda_1\right) + (1-\delta)\left(1-\lambda_2\right),
\end{aligned}$$

so that equation (A.30) holds. Hence, for any choice of δ, we can find a κ that ensures the equilibrium still holds. Note that there is nothing in this argument that is particular to $\tilde{m}_1^A$. As a result, we can construct examples of multiple equilibria of the form:

$$\mathbf{F}_1 = \begin{pmatrix}
\omega_1^A - \kappa_1^A \varepsilon_1^A; & \delta_1^A \varepsilon_1^A; & 1-\omega_1^A - \left(1-\kappa_1^A\right)\varepsilon_1^A; & \left(1-\delta_1^A\right)\varepsilon_1^A \\
\omega_1^B - \kappa_1^B \varepsilon_1^B; & \delta_1^B \varepsilon_1^B; & 1-\omega_1^B - \left(1-\kappa_1^B\right)\varepsilon_1^B; & \left(1-\delta_1^B\right)\varepsilon_1^B \\
\omega_1^C - \kappa_1^C \varepsilon_1^C; & \delta_1^C \varepsilon_1^C; & 1-\omega_1^C - \left(1-\kappa_1^C\right)\varepsilon_1^C; & \left(1-\delta_1^C\right)\varepsilon_1^C \\
\omega_1^D - \kappa_1^D \varepsilon_1^D; & \delta_1^D \varepsilon_1^D; & 1-\omega_1^D - \left(1-\kappa_1^D\right)\varepsilon_1^D; & \left(1-\delta_1^D\right)\varepsilon_1^D
\end{pmatrix}$$

$$\mathbf{F}_2 = \begin{pmatrix}
\delta_2^A \varepsilon_2^A; & \omega_2^A - \kappa_2^A \varepsilon_2^A; & \left(1-\delta_2^A\right)\varepsilon_2^A & 1-\omega_2^A - \left(1-\kappa_2^A\right)\varepsilon_2^A \\
\delta_2^B \varepsilon_2^B; & \omega_2^B - \kappa_2^B \varepsilon_2^B; & \left(1-\delta_2^B\right)\varepsilon_2^B & 1-\omega_2^B - \left(1-\kappa_2^B\right)\varepsilon_2^B \\
\delta_2^C \varepsilon_2^C; & \omega_2^C - \kappa_2^C \varepsilon_2^C; & \left(1-\delta_2^C\right)\varepsilon_2^C & 1-\omega_2^C - \left(1-\kappa_2^C\right)\varepsilon_2^C \\
\delta_2^D \varepsilon_2^D; & \omega_2^D - \kappa_2^D \varepsilon_2^D; & \left(1-\delta_2^D\right)\varepsilon_2^D & 1-\omega_2^D - \left(1-\kappa_2^D\right)\varepsilon_2^D
\end{pmatrix},$$

for many different chosen values of $\left\{\varepsilon_k^l\right\}$ and $\left\{\delta_k^l\right\}$. □

A.4 Proof of Proposition 4

The steady-state system of equations we would like to examine can be written as:

$$L_i W_i^{-\theta} = \left(\Omega^2\right)^{-\theta} \sum_j M_{ij} W_j^{\theta} \qquad \text{(A.31)}$$

$$W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{p}}\right) = \sum_{j \in S} B_i T_{ij} C_j W_j^{-(\sigma-1)\tilde{\sigma}}\left(L_j^{\frac{1}{p}}\right)^a, \qquad \text{(A.32)}$$

where $T_{ij} \equiv \tau_{ij}^{1-\sigma}$, $B_i \equiv \bar{A}_i^{(\sigma-1)\tilde{\sigma}} \bar{u}_i^{\tilde{\sigma}}$, $C_j \equiv \bar{A}_j^{\tilde{\sigma}\sigma} \bar{u}_j^{(\sigma-1)\tilde{\sigma}}$, $M_{ij} \equiv \mu_{ij}^{-\theta}$, $p \equiv (\tilde{\sigma}(1-(\alpha_1+\alpha_2)(\sigma-1)-\sigma(\beta_1+\beta_2)))^{-1}$, and $a \equiv \frac{(1+(\alpha_1+\alpha_2)\sigma+(\beta_1+\beta_2)(\sigma-1))}{(1-(\alpha_1+\alpha_2)(\sigma-1)-\sigma(\beta_1+\beta_2))}$. Note that $\frac{a-1}{p}\frac{1}{\sigma-1} = \rho \equiv (\alpha_1+\alpha_2+\beta_1+\beta_2)$.

In what follows, we will assume $\rho > \max\left(0, \frac{1}{\theta} - \frac{1}{\sigma-1}\right)$ and $a \in (0,1)$. In addition, we have the labor market clearing constraint $\sum_i L_i = \bar{L}$. Our goal is to provide bounds on Ω. Since $\Omega > 0$, it suffices (and is a little less notation-heavy) to find bounds on Ω^2. In what follows, we refer to equation (A.31) as the "migration equation" and equation (A.32) as the "trade equation".

Before continuing with the proof, we remind the reader of a number of helpful mathematical properties. Define $\|\mathbf{x}\|_p \equiv \left(\sum_i x_i^p\right)^{\frac{1}{p}}$. (With some abuse of terminology, we refer to $\|\mathbf{x}\|_p$ as the "p-norm of $\mathbf{x}$", even though it is formally a norm only if $p \geq 1$). First, we remind the reader of the relationship between different p norms. For any $0 < p < q$, we have the convenient relationship:

$$\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p . \tag{A.33}$$

More generally, for any $p < q$, we have:

$$N^{\frac{1}{q}-\frac{1}{p}} \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq C(p,q,\mu) N^{\frac{1}{q}-\frac{1}{p}} \|\mathbf{x}\|_p , \tag{A.34}$$

where $C(p,q,\mu) \equiv \left(\frac{p(\mu^q-\mu^p)}{(q-p)(\mu^p-1)}\right)^{\frac{1}{q}} \left(\frac{q(\mu^p-\mu^q)}{(p-q)(\mu^q-1)}\right)^{-\frac{1}{p}}$, and $\mu \geq \left(\frac{\max_i x_i}{\min_i x_i}\right)$. Note that $C(p,q,1) = 1$. The first inequality is the well known generalized mean inequality, whereas the second inequality is due to the less known result originally due to Specht (1960) and reprinted (in English) in the textbook by Mitrinovic and Vasic (1970) (see Theorem 1 on p. 79).

Second, recall the Cauchy–Schwarz inequality that for any $N \times 1$ vectors $\mathbf{x} \equiv [x_i]$ and $\mathbf{y} \equiv [y_i]$, we have:

$$\sum_i |x_i y_i| \leq \left(\sum_i x_i^2\right)^{\frac{1}{2}} \left(\sum_i y_i^2\right)^{\frac{1}{2}} \iff \|\{x_i y_i\}\|_1 \leq \|\{x_i\}\|_2 \|\{y_i\}\|_2 .$$

Third, recall that the matrix norm induced by the vector p-norm for square matrix $\mathbf{A}$ is defined as $\|\mathbf{A}\|_p \equiv \sup\left\{\frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p} | \mathbf{x} \neq 0\right\}$. Moreover, if $\mathbf{A}$ is real and symmetric then $\|\mathbf{A}\|_2 = \rho(\mathbf{A})$, i.e. the spectral radius of $\mathbf{A}$.

Fourth, recall that for any $N \times N$ real symmetric matrix $\mathbf{A}$ and any $N \times 1$ vector $\mathbf{x}$ we have the following quadratic form inequality:

$$\mathbf{x}'\mathbf{Ax} \leq \rho(\mathbf{A}) \|\mathbf{x}\|_2 ,$$

where the inequality is strict when $\mathbf{x}$ is the eigenvector associated with the largest eigenvalue.

A.4.1 Lemma

We now offer a lemma which provides a bound on the maximum of the ratio of the highest and lowest welfare and population across locations. This lemma is necessary in defining the constants c_1 and c_2 mentioned in the proposition.

Lemma 1. *In any steady-state equilibrium, we can bound the ratio of the maximum to minimum period ex-post welfare by:*

$$1 \leq \frac{\max_{i \in S} W_i}{\min_{i \in S} W_i} \leq \mu_W , \tag{A.35}$$

where:

$$\mu_W \equiv \max_{i,j} \left(\frac{\bar{u}_i}{\bar{u}_j}\right)^{\kappa_u} \left(\frac{\bar{A}_i}{\bar{A}_j}\right)^{\kappa_A} \left(\max_k \left(\frac{T_{ki}}{T_{kj}}\right)^{\kappa_P}\right) \left(\max_k \left(\frac{M_{ik}}{M_{jk}}\right)^{\kappa_\Pi}\right),$$

and $\kappa_u \equiv \frac{\sigma/\theta}{\gamma_2}$, $\kappa_A \equiv \frac{(\sigma-1)/\theta}{\gamma_2}$, $\kappa_P \equiv \frac{\tilde{\sigma}/\theta}{\gamma_2}$, $\kappa_\Pi \equiv \frac{\sigma/\theta^2}{\gamma_2} - \frac{1}{\theta}$, *and* $\gamma_2 \equiv 1 + \frac{\sigma}{\theta} - ((\alpha_1+\alpha_2)(\sigma-1) + (\beta_1+\beta_2)\sigma)$, *and recall* $\tilde{\sigma} \equiv \frac{2\sigma-1}{\sigma-1}$. *Similarly, we can bound the ratio of the maximum to minimum period ex-post welfare,* *inclusive of the idiosyncratic component, by:*

$$1 \leq \frac{\max W_i L_i^{-\frac{1}{\theta}}}{\min W_i L_i^{-\frac{1}{\theta}}} \leq \mu_{WL}, \tag{A.36}$$

where:

$$\mu_{WL} \equiv \max_{i,j,k} \left(\frac{M_{i,k}}{M_{j,k}} \right)^{\frac{1}{\theta}}.$$

Proof. The proof relies on two relationships that hold in steady-state. From the labor market clearing conditions, we have:

$$W_i^\theta \Pi_i^\theta \propto L_i, \tag{A.37}$$

where $\Pi_i^\theta \equiv \sum_k M_{ik} W_k^\theta$. Similarly, noting that $\Lambda_i^{-1} = \Pi_i$ in the steady-state, we have from equation (20):

$$\gamma \ln L_i = C_1 + \sigma \ln \bar{u}_i + (\sigma - 1) \ln \bar{A}_i - (2\sigma - 1) \ln P_i + \sigma \ln \Pi_i + (\alpha_2 (\sigma - 1) + \beta_2 \sigma) \ln L_i,$$

where, recall, $\gamma \equiv 1 + \frac{\sigma}{\theta} - (\alpha_1 (\sigma - 1) + \beta_1 \sigma)$. Combining these two equations and solving for $\ln W_i$ yields:

$$\ln W_i = \kappa_u \ln \bar{u}_i + \kappa_A \ln \bar{A}_i + \kappa_P \ln P_i^{1-\sigma} + \kappa_\Pi \ln \Pi_i^\theta + C, \tag{A.38}$$

for some constant C. We now show that we can bound the difference in Π_i^θ and $P_i^{1-\sigma}$ across locations. Note that for any i and j we have:

$$\frac{\Pi_i^\theta}{\Pi_j^\theta} = \frac{\sum_k M_{ik} W_k^\theta}{\sum_k M_{jk} W_k^\theta} = \sum_k \left(\frac{M_{jk} W_k^\theta}{\sum_l M_{jl} W_l^\theta} \right) \frac{M_{ik}}{M_{jk}}.$$

That is, $\frac{\Pi_i^\theta}{\Pi_j^\theta}$ is a weighted average of $\frac{M_{ik}}{M_{jk}}$, as $\sum_k \left(\frac{M_{jk} W_k^\theta}{\sum_l M_{jl} W_l^\theta} \right) = 1$. This in turn implies that for all i and j we have:

$$\min_k \frac{M_{ik}}{M_{jk}} \leq \frac{\Pi_i^\theta}{\Pi_j^\theta} \leq \max_k \frac{M_{ik}}{M_{jk}} \tag{A.39}$$

Similarly, for any i and j we have:

$$\min_k \frac{T_{ki}}{T_{kj}} \leq \frac{P_i^{1-\sigma}}{P_j^{1-\sigma}} \leq \max_k \frac{T_{ki}}{T_{kj}}. \tag{A.40}$$

Equation (A.38) implies:

$$\frac{\max_i W_i}{\min_j W_j} = \left(\frac{\bar{u}_{i^*_{max}}}{\bar{u}_{i^*_{min}}} \right)^{\kappa_u} \left(\frac{\bar{A}_{i^*_{max}}}{\bar{A}_{i^*_{min}}} \right)^{\kappa_A} \left(\frac{P_{i^*_{max}}^{1-\sigma}}{P_{i^*_{min}}^{1-\sigma}} \right)^{\kappa_P} \left(\frac{\Pi_{i^*_{max}}^\theta}{\Pi_{i^*_{min}}^\theta} \right)^{\kappa_\Pi},$$

where $i_W^{max} \equiv \arg\max_i W_i$ and $i_W^{min} \equiv \arg\min_i W_i$. Combining this with equations (A.39) and (A.40), we have:

$$\frac{\max_i W_i}{\min_j W_j} \leq \max_{i,j} \left(\left(\frac{\bar{u}_i}{\bar{u}_j} \right)^{\kappa_u} \left(\frac{\bar{A}_i}{\bar{A}_j} \right)^{\kappa_A} \left(\max_k \left(\frac{T_{ki}}{T_{kj}} \right)^{\kappa_p} \right) \left(\max_k \left(\frac{M_{ik}}{M_{jk}} \right)^{\kappa_\Pi} \right) \right) \equiv \mu_W,$$

as claimed.

Similarly, combining (the log of) equation (A.37) and applying the inequality from (A.39) yields:

$$\ln W_i L_i^{-\frac{1}{\theta}} = \ln \Pi_i + C_2 \implies$$

$$\frac{\max_i W_i L_i^{-\frac{1}{\theta}}}{\min_i W_i L_i^{-\frac{1}{\theta}}} \leq \left(\frac{\Pi_{i_{WL}^{max}}}{\Pi_{i_{WL}^{min}}} \right) \leq \left(\max_k \frac{M_{i_{WL}^{max} k}}{M_{i_{WL}^{min} k}} \right)^{\frac{1}{\theta}} = \max_{i,j} \left(\max_k \frac{M_{ik}}{M_{jk}} \right)^{\frac{1}{\theta}} \equiv \mu_{WL},$$

where $i_{WL}^{max} \equiv \arg\max_i W_i L_i^{-\frac{1}{\theta}}$ and $i_{WL}^{min} \equiv \arg\min_i W_i L_i^{-\frac{1}{\theta}}$, as required. $\square$

A.4.2 The upper bound

We now proceed by constructing the upper bound. The proof proceeds by first constructing an upper bound for steady-state welfare as a function of the norm of the period ex-post welfare using the migration equation. The proof then constructs an upper bound for the norm of the period ex-post welfare using the trade equation. Given the lemma above, combining the two results yields the stated bound immediately.

The migration equation: Using the fact that the quadratic form of a matrix is bounded above by the product of its largest eigenvalue and the norm of its eigenvector along with the labor market clearing condition, we have, starting with the migration equation (A.31):

$$\begin{aligned}
\left(\Omega^2\right)^{\theta} L_i W_i^{-\theta} = \sum_j M_{ij} W_j^{\theta} &\iff \\
\left(\Omega^2\right)^{\theta} L_i = W_i^{\theta} \sum_j M_{ij} W_j^{\theta} &\implies \\
\left(\Omega^2\right)^{\theta} \sum_i L_i = \sum_i W_i^{\theta} \sum_j M_{ij} W_j^{\theta} &\implies \\
\left(\Omega^2\right)^{\theta} \leq \frac{\rho\left(\mathbf{M}\right)}{\bar{L}} \times \left\|W_i^{\theta}\right\|_2 &\iff \\
\left(\Omega^2\right) \leq \left(\bar{L}\right)^{-\frac{1}{\theta}} \bar{\lambda}_M^{\frac{1}{\theta}} \left\|W\right\|_{2\theta}. &
\end{aligned} \tag{A.41}$$

where $\bar{\lambda}_M$ is the largest eigenvalue of $\mathbf{M}$. Because $\mathbf{M}$ is positive, this is also the spectral radius of $\mathbf{M}$.

The trade equation: Turning to the trade equation (A.32), we define $x_i \equiv C_i W_i^{-(\sigma-1)\tilde{\sigma}} \left(L_i^{\frac{1}{\rho}}\right)^a$ so that:

$$\begin{aligned}
W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{p}}\right) = \sum_{j\in S} B_i T_{ij} C_j W_j^{-(\sigma-1)\tilde{\sigma}} \left(L_j^{\frac{1}{p}}\right)^a &\iff \\
x_i = \frac{x_i B_i}{W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{\rho}}\right)} \sum_{j\in S} T_{ij} x_j &\iff \\
x_i = W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i \sum_{j\in S} T_{ij} x_j. &
\end{aligned}$$

We then sum both sides over i:

$$\begin{aligned}
x_i = W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i \sum_{j\in S} T_{ij} x_j &\implies \\
\sum_i x_i = \sum_i \left(W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i \sum_j T_{ij} x_j\right) &\implies \\
\sum_i x_i \leq \left(\sum_i \left(W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(\sum_j T_{ij} x_j\right)^2\right)^{\frac{1}{2}} &\implies \\
\left(\sum_i x_i^2\right)^{\frac{1}{2}} \leq \left(\sum_i \left(W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(\sum_j T_{ij} x_j\right)^2\right)^{\frac{1}{2}} &\implies
\end{aligned}$$

$$1 \leq \left(\sum_i \left(W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}} \sup_{\{x_i \geq 0\}} \frac{\left(\sum_i \left(\sum_j T_{ij} x_j\right)^2\right)^{\frac{1}{2}}}{\left(\sum_i x\right)^{\frac{1}{2}}} \iff$$
$$1 \leq \left(\sum_i \left(W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}} \rho\left(\mathbf{T}\right) \implies$$
$$1 \leq \left(\sum_i W_i^{1-\sigma} L_i^{\frac{a-1}{p}} B_i C_i\right) \rho\left(\mathbf{T}\right) \implies$$
$$1 \leq \left(\sum_i \left(W_i^{1-\sigma}\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}} \rho\left(\mathbf{T}\right),$$

where the third line uses the Cauchy-Schwartz inequality, the fourth line uses the fact that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$, the fifth line takes the supremum across all possible vectors $\mathbf{x}$, the sixth line notes that this is the definition of a matrix norm, the seventh line uses the fact (again) that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$, and the eighth line uses (again) the Cauchy-Schwartz inequality. Continuing, we have:

$$\left(\sum_i \left(W_i^{1-\sigma}\right)\right)^{-\frac{1}{\sigma-1}} \leq \left(\left(\sum_i \left(L_i^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{\sigma-1}} \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \iff$$
$$\|W\|_{2(1-\sigma)} \leq \left(\bar{L}^{\frac{a-1}{p}} \left(\sum_i \left(\left(\frac{L_i}{\bar{L}}\right)^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{\sigma-1}} \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \iff$$
$$\|W\|_{2(1-\sigma)} \leq \left(\bar{L}^{\frac{a-1}{p}} \left(\sum_i \left(\left(\frac{L_i}{\bar{L}}\right)^{\frac{a-1}{p}} B_i C_i\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{\sigma-1}} \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \iff$$
$$\|W\|_{2(1-\sigma)} \leq \bar{L}^{\rho} \times \left(\max_i B_i C_i\right)^{\frac{1}{\sigma-1}} \times \left(\sum_i \left(\left(\frac{L_i}{\bar{L}}\right)^{\frac{a-1}{p}}\right)^2\right)^{\frac{1}{2}\frac{1}{\sigma-1}} \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \iff$$
$$\|W\|_{2(1-\sigma)} \leq \bar{L}^{\rho} \times \max_i \bar{A}_i \bar{u}_i \times \left(\left\|\left(\frac{L_i}{\bar{L}}\right)\right\|_{2(\sigma-1)\rho}\right)^{\rho} \times \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \iff$$
$$\|W\|_{2(1-\sigma)} \leq \bar{L}^{\rho} \times \max_i \bar{A}_i \bar{u}_i \times \left(N^{\frac{1-2(\sigma-1)\rho}{2(\sigma-1)}}\right)^{\mathbf{1}\left\{\rho < \frac{1}{2(\sigma-1)}\right\}} \times \rho\left(\mathbf{T}\right)^{\frac{1}{\sigma-1}} \tag{A.42}$$

where the second line uses the fact that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ and rearranges, the third line uses the norm notation (and multiplies and divides by the aggregate labor), the fourth line bounds the effect of the local geography based on the best location, the fifth line rearranges, and the final line uses the relationships between different norms mentioned at the beginning of the proof (and aggregate labor market clearing).

The bound: Recall from equation (A.34) that because $(1-\sigma) < \theta$, we have:

$$\|\boldsymbol{W}\|_{2\theta} \leq c N^{\frac{1}{2}\left(\frac{1}{\theta}+\frac{1}{\sigma-1}\right)} \|\boldsymbol{W}\|_{2(1-\sigma)}, \tag{A.43}$$

where

$$c_1 \equiv C\left(2\left(1-\sigma\right), 2\theta, \mu_W\right) \tag{A.44}$$

from equation (A.34), and μ_W is defined above in equation (A.35) from Lemma 1. Combining equation (A.43) with the migration bound from equation (A.41) and the trade bound from equation (A.42) then yields:

$$\left(\Omega^2\right) \leq c \times \bar{\lambda}_M^{\frac{1}{\theta}} \times \bar{\lambda}_T^{\frac{1}{\sigma-1}} \times \max_i \bar{A}_i \bar{u}_i \times \bar{L}^{\rho-\frac{1}{\theta}} \times N^{\frac{1}{2}\left(\frac{1}{\theta}+\frac{1}{\sigma-1}\right)+\mathbf{1}\left\{\rho<\frac{1}{2(\sigma-1)}\right\}\frac{1-2(\sigma-1)\rho}{2(\sigma-1)}}, \tag{A.45}$$

as claimed.

A.4.3 The lower bound

We now proceed to prove the lower bound. As above, we first consider the migration equation and then consider the trade equation.

The migration equation: With some abuse of notation let M_{ij}^{-1} denote the $\{i,j\}^{th}$ element of $\mathbf{M}^{-1}$. Then:

$$\begin{aligned}
\left(\Omega^2\right)^{\theta} L_i W_i^{-\theta} &= \sum_j M_{ij} W_j^{\theta} \iff \\
\left(\Omega^2\right)^{-\theta} W_i^{\theta} &= \sum_j M_{ij}^{-1} L_j W_j^{-\theta} \iff \\
\left(\Omega^2\right)^{-\theta} L_i &= L_i W_i^{-\theta} \sum_j M_{ij}^{-1} L_j W_j^{-\theta} \implies \\
\left(\Omega^2\right)^{-\theta} \bar{L} &= \sum_i L_i W_i^{-\theta} \sum_j M_{ij}^{-1} L_j W_j^{-\theta} \implies \\
\left(\Omega^2\right)^{-\theta} \bar{L} &\leq \bar{\lambda}_{M^{-1}} \left\| L_i W_i^{-\theta} \right\|_2 \iff \\
\left(\Omega^2\right) &\geq \bar{L}^{\frac{1}{\theta}} \underline{\lambda}_M^{\frac{1}{\theta}} \left\| L_i^{-\frac{1}{\theta}} W_i \right\|_{-2\theta},
\end{aligned} \tag{A.46}$$

where the second line inverts the linear equation, the third line multiplies both sides by the population, the fourth line sums over all i and applies the labor market clearing condition, the fifth line uses the quadratic form inequality where $\bar{\lambda}_{M^{-1}}$ denotes the largest eigenvalue (in absolute value) of $\mathbf{M}^{-1}$, and the sixth line uses the fact that $\bar{\lambda}_{M^{-1}} = \left(\underline{\lambda}_M\right)^{-1}$, i.e. the largest eigenvalue (in absolute value) of $\mathbf{M}^{-1}$ is the inverse of the smallest eigenvalue of $\mathbf{M}$.

The trade equation: Again, with some abuse of notation, let T_{ij}^{-1} denote the $\{i,j\}^{th}$ element of $\mathbf{T}^{-1}$. Defining $y_i \equiv \frac{W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{\rho}}\right)}{B_i}$, inverting the linear system, and rearranging the trade equation (A.32) yields:

$$\begin{aligned}
W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{p}}\right) &= \sum_{j\in S} B_i T_{ij} C_j W_j^{-(\sigma-1)\tilde{\sigma}} \left(L_j^{\frac{1}{p}}\right)^a \iff \\
\frac{W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{p}}\right)}{B_i} &= \sum_{j\in S} T_{ij} C_j W_j^{-(\sigma-1)\tilde{\sigma}} \left(L_j^{\frac{1}{p}}\right)^a \iff
\end{aligned}$$

$$y_i = \frac{y_i}{C_i W_i^{-(\sigma-1)\tilde{\sigma}} \left(L_i^{\frac{1}{p}}\right)^a} \sum_{j \in S} T_{ij}^{-1} y_j \iff$$

$$y_i = \frac{\frac{W_i^{\tilde{\sigma}\sigma}\left(L_i^{\frac{1}{p}}\right)}{B_i}}{C_i W_i^{-(\sigma-1)\tilde{\sigma}} \left(L_i^{\frac{1}{p}}\right)^a} \sum_{j \in S} T_{ij}^{-1} y_j \iff$$

$$y_i = \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\frac{a-1}{p}\frac{1}{\sigma-1}-\frac{1}{\theta}\right)}}{B_i C_i}\right) \sum_{j \in S} T_{ij}^{-1} y_j.$$

Proceeding similarly to the upper bound discussed above, we have:

$$y_i = \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right) \sum_j T_{ij}^{-1} y_j \implies$$

$$\sum_i y_i = \sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right) \sum_j T_{ij}^{-1} y_j \implies$$

$$\sum_i y_i \leq \left(\sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(\sum_j T_{ij}^{-1} y_j\right)^2\right)^{\frac{1}{2}} \implies$$

$$\left(\sum_i y_i^2\right)^{\frac{1}{2}} \leq \left(\sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(\sum_j T_{ij}^{-1} y_j\right)^2\right)^{\frac{1}{2}} \implies$$

$$1 \leq \left(\sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right)^2\right)^{\frac{1}{2}} \sup_{\{y_i \geq 0\}} \frac{\left(\sum_i \left(\sum_j T_{ij}^{-1} y_j\right)^2\right)^{\frac{1}{2}}}{\left(\sum_i y_i^2\right)^{\frac{1}{2}}} \iff$$

$$\underline{\lambda}_T \leq \left(\sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right)^2\right)^{\frac{1}{2}} \implies$$

$$\underline{\lambda}_T \leq \sum_i \left(\frac{\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1} L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right) \implies$$

$$\underline{\lambda}_T \leq \left(\sum_i \left(\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1}\right)^2\right)^{\frac{1}{2}} \left(\sum_i \left(\frac{L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right)^2\right)^{\frac{1}{2}} \implies$$

$$\underline{\lambda}_T \leq \left(\sum_i \left(\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1}\right)^2\right)^{\frac{1}{2}} \sum_i \left(\frac{L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right) \implies$$

$$\underline{\lambda}_T \leq \left(\sum_i \left(\left(W_i L_i^{-\frac{1}{\theta}}\right)^{\sigma-1}\right)^2\right)^{\frac{1}{2}} \times \left(\min_i B_i C_i\right)^{-1} \times \sum_i \left(L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}\right), \tag{A.47}$$

where the second line sums over i, the third line applies the Cauchy-Schwartz inequality, the fourth line uses the fact that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$, the fifth line takes supremums, the sixth line combines the fact that the spectral radius is equal to the matrix 2-norm when the matrix is real and symmetric and the fact that the largest eigenvalue in absolute valueof $\mathbf{T}^{-1}$ is the smallest eigenvalue in magnitude of $\mathbf{T}$ (denoted by $\underline{\lambda}_T$), the seventh lines uses (again) the fact that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$, the eighth line uses (again) Cauchy-Schwartz, the ninth line uses (yet again) $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$, and the tenth lines uses the fact that $\sum_i \left(\frac{L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}}{B_i C_i}\right) \leq \left(\min_i B_i C_i\right)^{-1} \sum_i L_i^{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}$. Rearranging the last line yields:

$$\left\|W_i L_i^{-\frac{1}{\theta}}\right\|_{2(\sigma-1)} \geq \underline{\lambda}_T^{\frac{1}{\sigma-1}} \times \bar{L}^{\left(\rho-\frac{1}{\theta}\right)} \times \left(\min_i \bar{A}_i \bar{u}_i\right) \times \left(\left\|\left(\frac{L_i}{\bar{L}}\right)\right\|_{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}\right)^{\left(\rho-\frac{1}{\theta}\right)}.$$

Since we assume $\rho > \max\left(0, \frac{1}{\theta} - \frac{1}{\sigma-1}\right)$, we have $-(\sigma-1)\left(\rho-\frac{1}{\theta}\right) < 1$, so that from the norm inequalities referenced at the beginning of the proof:

$$\left\|\left(\frac{L_i}{\bar{L}}\right)\right\|_{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)} \leq N^{\frac{1}{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}-1} \left\|\left(\frac{L_i}{\bar{L}}\right)\right\|_1 \iff$$

$$\left\|\left(\frac{L_i}{\bar{L}}\right)\right\|_{-(\sigma-1)\left(\rho-\frac{1}{\theta}\right)} \leq N^{-\left(\frac{1}{(\sigma-1)\left(\rho-\frac{1}{\theta}\right)}+1\right)},$$

so that the inequality becomes:

$$\left\|W_i L_i^{-\frac{1}{\theta}}\right\|_{2(\sigma-1)} \geq \underline{\lambda}_T^{\frac{1}{\sigma-1}} \times \bar{L}^{\left(\rho-\frac{1}{\theta}\right)} \times \left(\min_i \bar{A}_i \bar{u}_i\right) \times N^{-\left(\rho+\frac{1}{\sigma-1}-\frac{1}{\theta}\right)}. \tag{A.48}$$

The bound: Recall from equation (A.34) that because $-\theta < (\sigma-1)$, we have:

$$N^{-\frac{1}{2}\left(\frac{1}{\sigma-1}+\frac{1}{\theta}\right)} c_2 \left\|W_i L_i^{-\frac{1}{\theta}}\right\|_{2(\sigma-1)} \leq \left\|W_i L_i^{-\frac{1}{\theta}}\right\|_{-2\theta}, \tag{A.49}$$

where

$$c_2 \equiv C\left(2\left(1-\sigma\right), 2\theta, \mu_{WL}\right)^{-1}, \tag{A.50}$$

from equation (A.34) and μ_{WL} is defined above in equation (A.36) from Lemma 1. Combining equation (A.46) from the migration bound with equation (A.48) from the trade bound then yields:

$$\left(\Omega^2\right) \geq c_2^{-1} \times \bar{L}^{\rho} \times \underline{\lambda}_M^{\frac{1}{\theta}} \times \underline{\lambda}_T^{\frac{1}{\sigma-1}} \times \left(\min_i \bar{A}_i \bar{u}_i\right) \times N^{-\left(\rho+\frac{1}{\sigma-1}+\frac{1}{2}\left(\frac{1}{\sigma-1}+\frac{1}{\theta}\right)\right)}, \tag{A.51}$$

where $\rho \equiv \alpha_1 + \alpha_2 + \beta_1 + \beta_2$, as claimed.

A.5 Proof of Proposition 5

Note that the four equations can be considered as two distinct systems of two equations, where the two systems of equations are:

$$\mathcal{P}_{it}^{1-\sigma} = \sum_j \widehat{T}_{ijt} \times Y_{jt} \times (P_{jt}^{1-\sigma})^{-1}$$
$$P_{it}^{1-\sigma} = \sum_j \widehat{T}_{jit} \times Y_{jt} \times (\mathcal{P}_{it}^{1-\sigma})^{-1},$$

and:

$$\left(\Lambda_{it}^{\theta}\right)^{-1} = \sum_j \widehat{M}_{jit} \times L_{jt-1} \times \left(\Pi_{jt}^{\theta}\right)^{-1}$$
$$\Pi_{it}^{\theta} = \sum_j \widehat{M}_{ijt} \times L_{jt} \times \Lambda_{jt}^{\theta}.$$

The first system of equations can be written as:

$$x_i = \sum_j K_{ij}^A y_j^{-1}$$
$$y_i = \sum_j K_{ij}^B x_j^{-1},$$

which has a corresponding LHS matrix of coefficients:

$$\boldsymbol{B} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the matrix on the RHS coefficients becomes:

$$\mathbf{\Gamma} \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Hence, we have:

$$\mathbf{A} \equiv \mathbf{\Gamma}\mathbf{B}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The second system of equations can be written as:

$$x_i^{-1} = \sum_j K_{ij}^A y_j^{-1}$$
$$y_i = \sum_j K_{ij}^B x_j,$$

which has a corresponding LHS matrix of coefficients:

$$\boldsymbol{B} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the matrix on the RHS coefficients becomes:

$$\mathbf{\Gamma} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, we have:

$$\mathbf{A} \equiv \mathbf{\Gamma}\mathbf{B}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In both systems, we have $\mathbf{A}^p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is then straightforward to check that $\rho\left(\mathbf{A}^p\right) = 1$, as required.

B Online Appendix: Possible Microfoundations for Spillovers

Section 2 briefly discussed several microfoundations for the productivity and amenity spillover functions in equations (1) and (3), respectively. This appendix elaborates.

B.1 Productivity spillovers

We formalize two models—based on the persistence of local knowledge and the durability of investments, in turn—that provide examples of formal microfoundations for the productivity spillover function, $A_{it} = \bar{A}_{it} L_{it}^{\alpha_1} L_{it-1}^{\alpha_2}$.

B.1.1 Microfoundation #1: persistence of local knowledge

We follow Deneckere and Judd (1992). Suppose that firms can pay a fixed cost f_i (in terms of local labor) to create a new variety, over which they have monopoly rights for one period (the period in which they introduce the variety). In the subsequent period, the new variety exists but is produced under conditions of perfect competition. In the following period (two periods after its introduction), we assume the variety no longer exists (i.e. its value to consumers has fully depreciated). Finally, we assume that consumers have Cobb-Douglas preferences (within locations) over the the new varieties and the old varieties, and CES preferences across respectively.

Demand: Let Ω_{it}^{new} be the set of varieties created by monopolistically competitive firms in period t in location i and $\Omega_{i,t}^{old}$ be the set of varieties created in the previous period that are now produced under perfect competition. We assume that consumers have the following preferences:

$$C_{jt} = \left(\sum_i \left(\left(\left(\int_{\Omega_{it}^{new}} q_{ijt}(\omega)^{\frac{\rho-1}{\rho}} d\omega \right)^{\frac{\rho}{\rho-1}} \right)^{\chi} \left(\left(\int_{\Omega_{it}^{old}} q_{ijt}(\omega)^{\frac{\rho-1}{\rho}} d\omega \right)^{\frac{\rho}{\rho-1}} \right)^{1-\chi} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where $q_{ijt}(\omega)$ is the quantity consumed in country j of variety ω from location i. Hence, ρ is the elasticity of substitution between varieties of a given type from a given location, χ is the Cobb-Douglas share of the CES composite of new varieties from a given location, and σ is the elasticity of substitution of the aggregate bundles (of new and old goods) across locations.

Given these preferences, the quantity a consumer in location j in period t will demand from firm ω in location i can be written as:

$$q_{ijt}(\omega) = \begin{cases} \chi p_{ijt}(\omega)^{-\rho} \left(P_{it}^{new}\right)^{\rho-1} \times \dfrac{\tau_{ijt}^{1-\sigma}\left(\left(P_{it}^{new}\right)^{\chi}\left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma}}{\sum_k \tau_{ijt}^{1-\sigma}\left(\left(P_{kt}^{new}\right)^{\chi}\left(P_{kt}^{old}\right)^{1-\chi}\right)^{1-\sigma}} Y_{jt} & \text{if } \omega \in \Omega_{it}^{new} \\ (1-\chi) p_{ijt}(\omega)^{-\rho} \left(P_{it}^{old}\right)^{\rho-1} \times \dfrac{\tau_{ijt}^{1-\sigma}\left(\left(P_{it}^{new}\right)^{\chi}\left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma}}{\sum_k \tau_{ijt}^{1-\sigma}\left(\left(P_{kt}^{new}\right)^{\chi}\left(P_{kt}^{old}\right)^{1-\chi}\right)^{1-\sigma}} Y_{jt} & \text{if } \omega \in \Omega_{it}^{old}, \end{cases} \tag{B.1}$$

where:

$$\left(P_{it}^{new}\right)^{1-\rho} \equiv \int_{\Omega_{it}^{new}} p_{ijt}(\omega)^{1-\rho} d\omega \tag{B.2}$$

$$\left(P_{it}^{old}\right)^{1-\rho} \equiv \int_{\Omega_{it}^{old}} p_{ijt}(\omega)^{1-\rho} d\omega \tag{B.3}$$

denote the price indices of the inner CES nests.

Supply: Let $c_{it} \equiv \frac{w_{it}}{\bar{A}_{it}}$ denote the marginal cost of production by a firm, where $\bar{A}_{it}$ is the (exogenous) productivity. The optimization problem faced by firm ω is:

$$\max_{\{q_{ijt}(\omega)\}_j} \sum_j \left(p_{ijt}(\omega)\, q_{ijt}(\omega) - c_{it}\tau_{ijt} q_{ijt}(\omega)\right) - w_{it} f_{it},$$

subject to consumer demand given by equation (B.1).

As a result, conditional on positive production (of which more below), the first order conditions imply:

$$p_{ijt}(\omega) = \frac{\rho}{\rho - 1} c_{it}\tau_{ijt}, \tag{B.4}$$

so that the price index across new varieties within a location is:

$$P_{it}^{new} \equiv \left(M_{it}^{new}\right)^{\frac{1}{1-\rho}} \left(\frac{\rho}{\rho - 1} c_{it}\right). \tag{B.5}$$

Profits of monopolistically competitive firms: The profits of a firm $\omega \in \Omega_{it}^{new}$ are:

$$\pi_{it}(\omega) \equiv \sum_j \left(p_{ijt}(\omega) - c_{it}\tau_{ijt}\right) q_{ijt}(\omega) - w_{it} f_{it}. \tag{B.6}$$

Substituting the consumer demand expression (B.1) and the price expression (B.4) into equation (B.6) yields:

$$\pi_{it}(\omega) = \chi \frac{1}{\rho} \left(\frac{\rho}{\rho - 1}\right)^{1-\rho} \sum_j \left(c_{it}\tau_{ijt}\right)^{1-\rho} \left(P_{it}^{new}\right)^{\rho-1} \frac{\tau_{ijt}^{1-\sigma} \left(\left(P_{it}^{new}\right)^{\chi} \left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma}}{\sum_k \tau_{ijt}^{1-\sigma} \left(\left(P_{kt}^{new}\right)^{\chi} \left(P_{kt}^{old}\right)^{1-\chi}\right)^{1-\sigma}} Y_{jt} - w_{it} f_{it}$$

Noting that, from the consumer demand equation (B.1) and the price expression (B.4), the revenue a producer receives is:

$$r_{it}(\omega) \equiv \sum_j p_{ijt}(\omega)\, q_{ijt}(\omega) \iff$$

$$r_{it}(\omega) \left(\frac{\rho}{\rho - 1}\right)^{\rho-1} \frac{1}{\chi} = \sum_j \left(c_{it}\tau_{ijt}\right)^{1-\rho} \left(P_{it}^{new}\right)^{\rho-1} \frac{\tau_{ijt}^{1-\sigma} \left(\left(P_{it}^{new}\right)^{\chi} \left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma}}{\sum_k \tau_{ijt}^{1-\sigma} \left(\left(P_{kt}^{new}\right)^{\chi} \left(P_{kt}^{old}\right)^{1-\chi}\right)^{1-\sigma}} Y_{jt}, \tag{B.7}$$

it is apparent that variable profits are simply equal to revenue divided by the elasticity of substitution:

$$\pi_{it}(\omega) + w_{it} f_{it} = \frac{1}{\rho} r_{it}(\omega). \tag{B.8}$$

Free entry: From the free entry condition, total profits of a firm are zero, i.e. $\pi_{it}(\omega) = 0$. Applying the free entry condition to equation (B.8) yields:

$$w_{it} f_{it} = \frac{1}{\rho} r_{it}(\omega) \tag{B.9}$$

Substituting equation (B.9) into equation (B.7) yields:

$$\sum_j \tau_{ijt}^{1-\rho} w_{it}^{-\rho} A_{it}^{\rho-1} \left(P_{it}^{new}\right)^{\rho-1} \frac{\tau_{ijt}^{1-\sigma} \left(\left(P_{it}^{new}\right)^{\chi} \left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma}}{\sum_k \tau_{ijt}^{1-\sigma} \left(\left(P_{kt}^{new}\right)^{\chi} \left(P_{kt}^{old}\right)^{1-\chi}\right)^{1-\sigma}} Y_{jt} = \frac{1}{\chi} \left(\frac{\rho}{\rho - 1}\right)^{\rho-1} \rho f_{it}, \tag{B.10}$$

where we use the fact that $c_{it} = w_{it}/A_{it}$.

Perfectly competitive varieties: The price charged for the perfectly competitive varieties $\omega \in \Omega_{it}^{new}$ is simply the marginal cost:

$$p_{ijt}(\omega) = \tau_{ijt} c_{it} \ \forall \omega \in \Omega_{it}^{new},$$

so that:

$$P_{it}^{old} = \left(M_{it}^{old}\right)^{\frac{1}{1-\rho}} c_{it}, \tag{B.11}$$

where $M_{it}^{old} \equiv \left|\Omega_{it}^{old}\right|$ denotes the measure of existing varieties.

Labor market clearing: Labor market clearing requires that the total labor used by all firms (for entry and production of the new varieties as well as production of the existing varieties) must equal to the total number of workers in the location, $L_{i,t}$. The total amount of labor required by new varieties is:

$$\begin{aligned} L_{it}^{new} &= \int\limits_{\Omega_{it}^{new}} \left(\sum_j \tau_{ijt} \frac{q_{ijt}(\omega)}{\bar{A}_{it}} + f_i\right) d\omega \iff \\ L_{it}^{new} &= \rho f_{it} M_{it}^{new}, \end{aligned}$$

where $M_{it}^{new} \equiv |\Omega_{it}^{new}|$ denotes the measure of new varieties and we have used the free entry equation (B.10). Similarly, the total amount of labor required by old varieties is:

$$\begin{aligned} L_{it}^{old} &= \int\limits_{\Omega_{it}^{old}} \left(\sum_j \tau_{ijt} \frac{q_{ijt}(\omega)}{\bar{A}_{it}}\right) d\omega \iff \\ L_{it}^{old} &= M_{it}^{new} \frac{1-\chi}{\chi} \rho f_{it}, \end{aligned}$$

where we have used the equations for the old and new variety price indices from equations (B.5) and (B.11).

Total labor used by all firms is hence:

$$\begin{aligned} L_{it}^{new} + L_{it}^{old} = L_{it} &\iff \\ M_{it}^{new} &= \chi \frac{L_{it}}{\rho f_{it}}, \end{aligned} \tag{B.12}$$

so that the measure of new firms is proportional to the labor supply.

The productivity microfoundation: Combining the old and new variety price indices from equations (B.5) and (B.11) yields:

$$\left(\left(P_{it}^{new}\right)^{\chi} \left(P_{it}^{old}\right)^{1-\chi}\right)^{1-\sigma} = \left(c_{it}\right)^{1-\sigma} \frac{\rho}{\rho-1}^{(1-\sigma)\chi} \left(M_{it}^{new}\right)^{\chi\left(\frac{1-\sigma}{1-\rho}\right)} \left(M_{it}^{old}\right)^{(1-\chi)\left(\frac{1-\sigma}{1-\rho}\right)}.$$

Total trade flows from i to j at time t are determined by simply aggregating across all firms of both types. The total trade of new varieties is thus:

$$\begin{aligned} X_{ijt}^{new} &= \int_{\Omega_{it}^{new}} p_{ijt}(\omega) q_{ijt}(\omega) d\omega \iff \\ X_{ijt}^{new} &= \chi \frac{\left(\tau_{ijt} c_{it}\right)^{1-\sigma} \left(M_{it}^{new}\right)^{\chi\left(\frac{1-\sigma}{1-\rho}\right)} \left(M_{it}^{old}\right)^{(1-\chi)\left(\frac{1-\sigma}{1-\rho}\right)}}{\sum_k \left(\tau_{kjt} c_{kt}\right)^{1-\sigma} \left(M_{kt}^{new}\right)^{\chi\left(\frac{1-\sigma}{1-\rho}\right)} \left(M_{kt}^{old}\right)^{(1-\chi)\left(\frac{1-\sigma}{1-\rho}\right)}} Y_{jt}. \end{aligned}$$

Similarly, the total trade of existing varieties is:

$$X_{ijt}^{old} = \int_{\Omega_{it}^{old}} p_{ijt}(\omega) q_{ijt}(\omega) d\omega \iff$$
$$X_{ijt}^{old} = (1-\chi) \frac{(\tau_{ijt}c_{it})^{1-\sigma} (M_{it}^{new})^{\chi\left(\frac{1-\sigma}{1-\rho}\right)} \left(M_{it}^{old}\right)^{(1-\chi)\left(\frac{1-\sigma}{1-\rho}\right)}}{\sum_k (\tau_{kjt}c_{kt})^{1-\sigma} (M_{kt}^{new})^{\chi\left(\frac{1-\sigma}{1-\rho}\right)} \left(M_{kt}^{old}\right)^{(1-\chi)\left(\frac{1-\sigma}{1-\rho}\right)}} Y_{jt}.$$

Hence, total trade flows are:

$$X_{ijt} = X_{ijt}^{new} + X_{ijt}^{old} \iff$$
$$X_{ijt} = \tau_{ijt}^{1-\sigma} w_{it}^{1-\sigma} A_{it}^{\sigma-1} P_{jt}^{\sigma-1} Y_{jt},$$

where:

$$P_{jt}^{1-\sigma} \equiv \sum_k \tau_{kjt}^{1-\sigma} w_{kt}^{1-\sigma} A_{kt}^{\sigma-1}$$

and:

$$A_{it} \equiv \bar{A}_{it} f_{it}^{\frac{1}{\rho-1}} \times L_{it}^{\alpha_1} \times L_{it-1}^{\alpha_2},$$

and $\alpha_1 \equiv \frac{\chi}{\rho-1}$ and $\alpha_2 \equiv \frac{1-\chi}{\rho-1}$, as claimed.

B.1.2 Microfoundation #2: durable investments in local productivity

Setup: In each location i, there is a measure of firms that compete a la Bertrand. Firms can hire workers either to produce or to innovate, where the total quantity produced in location i at time t depends on the amount of labor used in the production L_{it}, the amount of land H_{it}, the amount of innovation ϕ_{it} and some productivity shifter B_{it}:

$$Q_{it} = \phi_{it}^{\gamma_1} B_{it} L_{it}^{\mu} H_{it}^{1-\mu} \iff$$
$$q_{it} = \phi_{it}^{\gamma_1} B_{it} l_{it}^{\mu},$$

where in what follows we focus on the output per unit land q_{it} and the labor per unit land l_{it}. We assume the parameters satisfy $\mu < 1$ (due to the diminishing marginal product of labor per unit land) and $\gamma_1 < 1$ (due to the diminishing marginal product of innovation).

To employ a level of innovation ϕ_{it}, a firm must hire $\nu\phi_{it}^{\xi}$ additional units of labor, where $\xi < \gamma_1/(1-\mu)$. We assume that innovation today has an affect on the level of productivity tomorrow so that:

$$B_{it} = \phi_{it-1}^{\delta\gamma_1} \bar{B}_{it}, \tag{B.13}$$

where $\bar{B}_{it}$ is an exogenous shock and $\delta < 1$ indicates the extent to which innovation decays from one period to the next. We assume the cost per unit of land r_{it} is determined by a competitive auction, so that firms obtain zero profits.

Profit maximization: Even though innovations today affect innovations in future periods, because firms earn zero profits in the future, the dynamic problem reduces to a sequence of static profit maximizing problems Desmet and Rossi-Hansberg (2014).

As a result the firms' profit maximization problem becomes:

$$\max_{l_{it},\phi_{it}} p_{it} B_{it} (\phi_{it}^{\gamma_1}) \times (l_{it}^{\mu}) - w_{it} \underbrace{l_{it}}_{\#\text{ of production workers}} - w_{it} \underbrace{\left(\nu\phi_{it}^{\xi}\right)}_{\#\text{ of innovation workers}} - r_{it},$$

which has the following first order conditions:

$$\gamma_1 B_{it} p_{it} \phi_{it}^{\gamma_1 - 1} l_{it}^{\mu} = \xi \nu w_{it} \phi_{it}^{\xi - 1}$$
$$\mu B_{it} p_{it} \phi_{it}^{\gamma_1} l_{it}^{\mu - 1} = w_{it},$$

which combine to yield:

$$\frac{\gamma_1}{\mu} l_{it} = \xi \nu \phi_{it}^{\xi} \iff$$
$$\left(\frac{\gamma_1}{\mu \xi \nu} l_{it} \right)^{\frac{1}{\xi}} = \phi_{it}. \tag{B.14}$$

Total employment $\tilde{l}_{it}$ per unit land is equal to the sum of the production workers and the innovation workers:

$$\tilde{l}_{it} = l_{it} + \nu \phi_{it}^{\xi} \iff$$
$$\tilde{l}_{it} = \left(1 + \frac{\gamma_1}{\mu \xi} \right) l_{it}.$$

Rent and income:

Equilibrium rent ensures that profits per unit land are equal to zero:

$$r_{it} = B_{it} p_{it} \phi_{it}^{\gamma_1} l_{it}^{\mu} + w_{it} l_{it} + \nu w_{it} \phi_{it}^{\xi} \iff$$
$$r_{it} = \left(\frac{1}{\mu} + 1 + \frac{\gamma_1}{\mu \xi} \right) w_{it} l_{it}.$$

Note that total income per unit labor in a location is:

$$Y_{it} = r_{it} H_{it} + w_{it} \tilde{L}_{it} \iff$$
$$\frac{Y_{it}}{\tilde{L}_{it}} = \left(\frac{\frac{1}{\mu} + 1 + \frac{\gamma_1}{\mu \xi}}{\left(1 + \frac{\gamma_1}{\mu \xi} \right)} + 1 \right) w_{it}.$$

The productivity microfoundation:

The output price is:

$$\mu B_{it} p_{it} \phi_{it}^{\gamma_1} L_{it}^{\mu - 1} = w_{it} \iff$$
$$p_{it} = \frac{1}{B_{it}} \left(\frac{1}{\mu} \left(\frac{\xi \nu \mu}{\gamma_1} \right)^{\frac{\gamma_1}{\xi}} \right) w_{it} l_{it}^{1 - \mu - \frac{\gamma_1}{\xi}}$$

total output is:

$$q_{it} = \phi_{it}^{\gamma_1} B_{it} l_{it}^{\mu} \iff$$
$$Q_{it} = \left(\frac{\gamma_1}{\mu \xi \nu} \right)^{\frac{\gamma_1}{\xi}} B_{it} \tilde{L}_{it}^{\mu + \frac{\gamma_1}{\xi}} H_{it}^{1 - \mu - \frac{\gamma_1}{\xi}},$$

where $\tilde{L}_{it}$ is total employment in location i at time t. Combining equations (B.13) and (B.14) yields:

$$B_{it} = \phi_{it-1}^{\delta \gamma_1} \bar{B}_{it} \iff$$
$$B_{it} = \left(\frac{\frac{\gamma_1}{\mu \xi \nu}}{\left(1 + \frac{\gamma_1}{\mu \xi} \right)} \frac{\tilde{L}_{it-1}}{H_{it-1}} \right)^{\delta \frac{\gamma_1}{\xi}} \bar{B}_{it},$$

so that in total we have:

$$Q_{it} = \left(\frac{\gamma_1}{\mu\xi\nu}\right)^{\frac{\gamma_1}{\xi}} \left(\left(\frac{\frac{\gamma_1}{\mu\xi\nu}}{\left(1+\frac{\gamma_1}{\mu\xi}\right)}\frac{\tilde{L}_{it-1}}{H_{it-1}}\right)^{\delta\frac{\gamma_1}{\xi}} \bar{B}_{it}\right) \tilde{L}_{it}^{\mu+\frac{\gamma_1}{\xi}} H_{it}^{1-\mu-\frac{\gamma_1}{\xi}} \iff$$
$$Q_{it} = \bar{A}_{it}\tilde{L}_{it}^{\alpha_1}\tilde{L}_{it-1}^{\alpha_2}\tilde{L}_{it},$$

where $\bar{A}_{it} \equiv \left(\frac{\gamma_1}{\mu\xi\nu}\right)^{(1+\delta)\frac{\gamma_1}{\xi}} \left(1+\frac{\gamma_1}{\mu\xi}\right)^{-\delta\frac{\gamma_1}{\xi}} \bar{B}_{it}H_{it}^{1-\mu-\frac{\gamma_1}{\xi}} H_{it-1}^{-\delta\frac{\gamma_1}{\xi}}$, $\alpha_1 \equiv \frac{\gamma_1}{\xi} - (1-\mu)$, and $\alpha_2 \equiv \delta\frac{\gamma_1}{\xi}$, as required.

B.2 Amenity spillover

We formalize here a possible microfoundation for the amenity spillover function, $u_{it} = \bar{u}_{it}L_{it}^{\beta_1}L_{it-1}^{\beta_2}$.

Demand: Suppose that consumers have Cobb-Douglas preferences over land and a consumption good, so that their indirect utility function can be written as:

$$W_{it} = \frac{(Y_{it}/L_{it})}{(P_{it})^{\lambda}\left(r_{it}^{H}\right)^{1-\lambda}},$$

where r_{it}^{H} is the rental cost of housing. Let H_{it} denote the (equilibrium quantity) of housing and let K_{it} denote the (exogenous) quantity of land in a location, so that $h_{it} \equiv H_{it}/K_{it}$ is the housing density (e.g. square feet of housing per acre of land).

Given the Cobb-Douglas preferences (and, from balanced trade, that income equals expenditure, $Y_{it} = E_{it}$), we have:

$$r_{it}^{H}H_{it} = (1-\lambda)\,Y_{it}$$
$$w_{it}L_{it} = \lambda Y_{it},$$

so that we can write the payment to housing as a function of the payment to labor:

$$r_{it}^{H} = \left(\frac{1-\lambda}{\lambda}\right)\frac{1}{H_{it}}w_{it}L_{it}.$$

Note then that we can write:

$$W_{it} = \frac{(Y_{it}/L_{it})}{(P_{it})^{\lambda}\left(r_{it}^{H}\right)^{1-\lambda}} \iff$$
$$\tilde{W}_{it} = \frac{1}{\lambda\,(1-\lambda)^{\frac{1-\lambda}{\lambda}}}\frac{w_{it}}{P_{it}}\left(\frac{H_{it}}{L_{it}}\right)^{\frac{1-\lambda}{\lambda}}, \tag{B.15}$$

where $\tilde{W}_{it} \equiv W_{it}^{\frac{1}{\lambda}}$ is a positive monotonic transform of W_{it} that hence can serve as our measure of welfare.

Supply: We now determine the equilibrium stock of housing H_{it}. Suppose that each unit of land is owned by a representative developer, who decides how much to upgrade the housing tract. The amount of housing per unit land ($h_{it} \equiv \frac{H_{it}}{K_{it}}$) is a function of the housing stock that has survived from the previous period $\left(h_{it}^{existing} \equiv \frac{H_{it}^{existing}}{K_{it}}\right)$ and the amount of labor that the firm chooses to hire to rebuild it:

$$h_{it} = \left(h_{it}^{existing}\right)^{\mu}\left(l_{it}^{d}\right)^{1-\mu} \iff$$
$$H_{it} = \left(H_{it}^{existing}\right)^{\mu}\left(L_{it}^{d}\right)^{1-\mu}.$$

In what follows, we assume for simplicity that the existing housing stock from period $t-1$ in period t is some fraction of the development in the previous period:

$$H_{it}^{existing} = \bar{C}_{it} \left(L_{it-1}^{d} \right)^{\rho}, \tag{B.16}$$

where $\bar{C}_{it}$ is an (exogenous) shock.

Profit maximization: A developer solves:

$$\max_{l_{it}^d} r_{it}^H h_{it} - w_{it} l_{it}^d - f_{it} \iff$$
$$\max_{l_{it}^d} r_{it}^H \left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{1-\mu} - w_{it} l_{it}^d - f_{it},$$

where f_{it} is a fixed cost (a "permit cost") that is remitted back to local residents and is set via a competitive biding process, ensuring that the firm earns zero profits (and hence the dynamic problem simplifies into a series of static profit maximization problems, as above).

First order conditions are:

$$(1-\mu) r_{it}^H \left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{-\mu} = w_{it} \iff$$
$$\left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{1-\mu} = \frac{1}{1-\mu} \frac{1}{r_{it}^H} w_{it} l_{it}^d.$$

Note that the fixed "permit costs" are then:

$$f_{it} = r_{it}^H \left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{1-\mu} - w_{it} l_{it}^d \iff$$
$$f_{it} = \left(\frac{\mu}{1-\mu} \right) w_{it} l_{it}^d,$$

which recall are remitted to workers and ensure profits are zero.

We can combine this with the rental rate above to calculate the fraction of workers hired in the development of the land:

$$h_{it} = \left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{1-\mu} \iff$$
$$(1-\mu) \left(\frac{1-\lambda}{\lambda} \right) L_{it} = L_{it}^d,$$

so we require as a parametric restriction (so that only a fraction of workers are hired as local developers):

$$(1-\mu) \left(\frac{1-\lambda}{\lambda} \right) < 1.$$

Since a constant fraction of local workers are hired, we can express the housing density solely as a function of the local population, the local land area, and then:

$$h_{it} = \left(h_{it}^{existing} \right)^{\mu} \left(l_{it}^d \right)^{1-\mu} \iff$$
$$H_{it} = \left((1-\mu) \left(\frac{1-\lambda}{\lambda} \right) \right)^{(1-\mu)+\rho\mu} \bar{C}_{it}^{\mu} \left(L_{it-1} \right)^{\rho\mu} \left(L_{it} \right)^{1-\mu}. \tag{B.17}$$

The amenity microfoundation: We substitute equation (B.17) for the equilibrium stock of housing into the welfare equation (B.15) to yield:

$$\tilde{W}_{it} = \frac{1}{\lambda\left(1-\lambda\right)^{\frac{1-\lambda}{\lambda}}} \frac{w_{it}}{P_{it}} \left(\frac{H_{it}}{L_{it}}\right)^{\frac{1-\lambda}{\lambda}} \iff$$

$$\tilde{W}_{it} = \frac{w_{it}}{P_{it}} \bar{u}_{it} L_{it}^{\beta_1} L_{it-1}^{\beta_2},$$

where $\bar{u}_{it} \equiv \frac{1}{\lambda(1-\lambda)^{\frac{1-\lambda}{\lambda}}} \left((1-\mu)\left(\frac{1-\lambda}{\lambda}\right)\right)^{\frac{1-\lambda}{\lambda}((1-\mu)+\rho\mu)} \bar{C}_{it}^{\frac{1-\lambda}{\lambda}}$, $\beta_1 \equiv -\mu\frac{1-\lambda}{\lambda}$, and $\beta_2 \equiv \rho\mu\frac{1-\lambda}{\lambda}$ as required.

C Online Appendix: Additional tables and figures

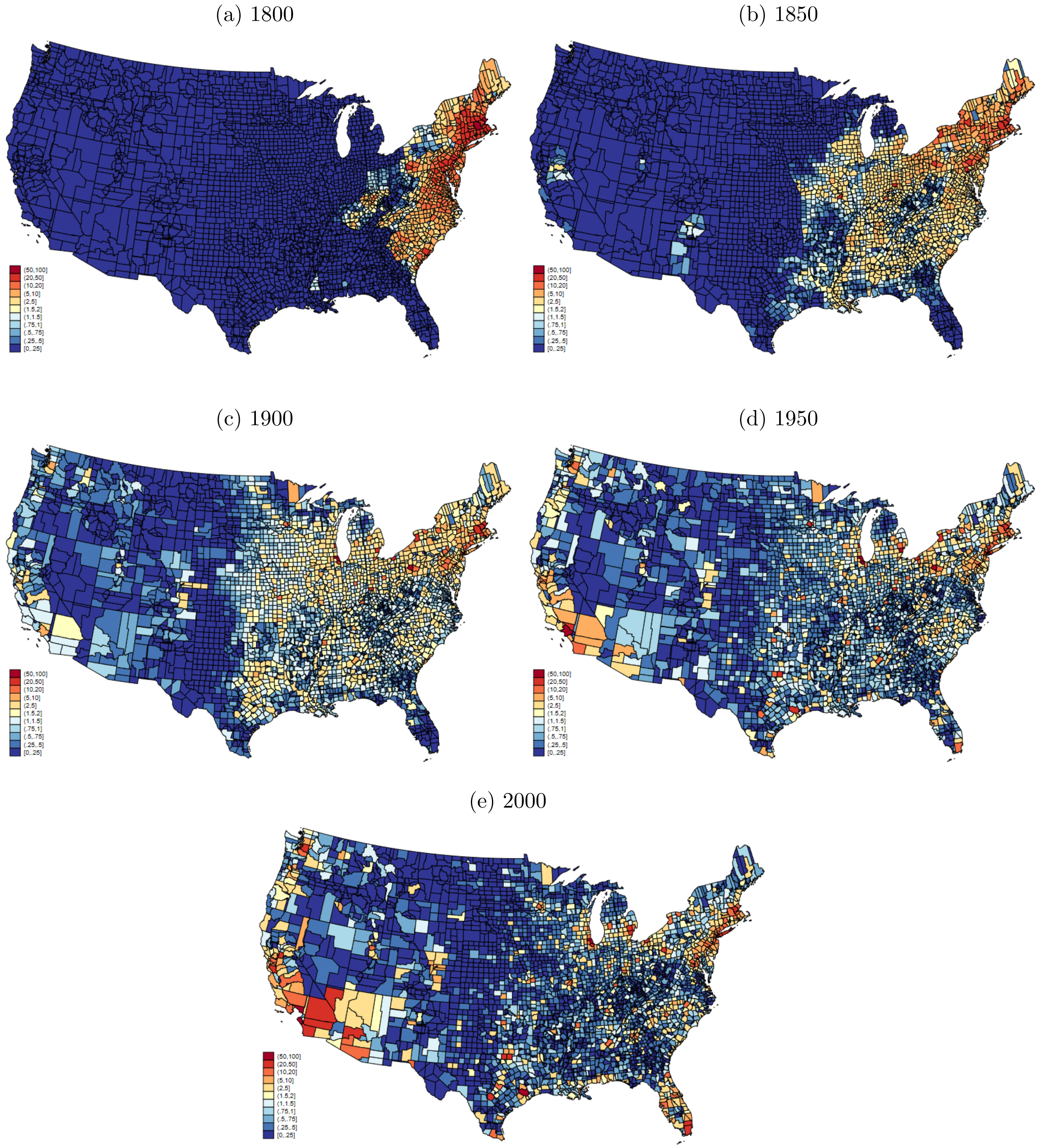
This section includes additional tables and figures mentioned in footnotes in the text.

Table C.1: First-stage estimates

	$\ln(L_{it})$	$\ln(L_{it-1})$	$\ln(P_{it}^{1-\sigma})$	$\ln(W_{it}^{\theta})$
	(1)	(2)	(3)	(4)
Instruments shifting amenities (used to estimate productivity spillovers):				
Year*(Average max. temp. in hottest month)	5.923***	-11.821***	0.079	8.499***
	(1.345)	(2.734)	(0.171)	(1.346)
Year*(Average max. temp. in hottest month)^2	-0.011***	0.024***	-0.000	-0.017***
	(0.002)	(0.005)	(0.000)	(0.002)
Year*(Average min. temp. in coldest month)	1.542***	-0.406*	0.324***	1.817***
	(0.132)	(0.245)	(0.019)	(0.132)
Year*(Average min. temp. in coldest month)^2	0.010***	0.010***	0.002***	0.008***
	(0.001)	(0.001)	(0.000)	(0.001)
Instruments shifting productivities (used to estimate amenity spillovers):				
Year*(High - low inten. corn potential yield, mean)	0.026***	0.033***	0.003***	0.024***
	(0.003)	(0.006)	(0.000)	(0.003)
Year*(High inten. soy - low inten. wheat potential yield, mean)	-0.048***	-0.034***	-0.004***	-0.046***
	(0.007)	(0.011)	(0.001)	(0.007)
Year*(High - low inten. corn potential yield, st. dev.)	-0.004	-0.019	0.003**	0.002
	(0.007)	(0.014)	(0.001)	(0.007)
Year*(High inten. soy - low inten. wheat potential yield, st. dev.)	0.001	0.072**	-0.002	-0.005
	(0.017)	(0.031)	(0.003)	(0.017)
F-statistic	24.995	36.854	48.109	26.343
R-squared	0.890	0.763	0.965	0.861
Observations	15,640	15,640	15,640	15,640

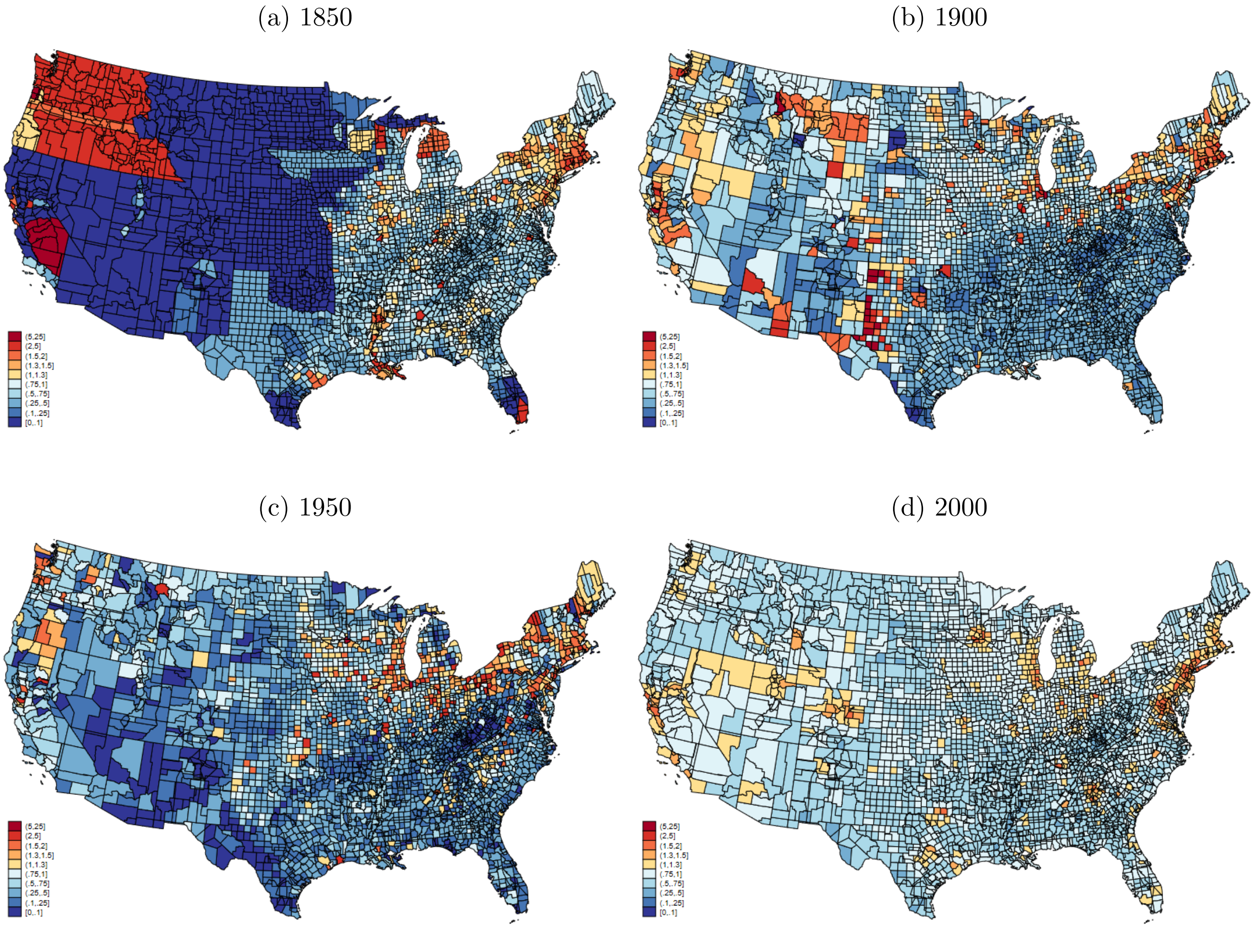
Notes: This table reports OLS regressions of the first-stage equations corresponding to the 2SLS estimates of productivity and amenity spillover functions reported in Tables 2 and 3.. Each observation is a sub-county from 1850-2000. All specifications control for sub-county fixed effects and region-year fixed effects, where a region is one of 14 equally sized squares covering the continental U.S.. The sample is all sub-counties in all years where geographic instruments and contemporaneous/lagged population values are observed. Standard errors are two-way clustered at the sub-county (to allow for serial correlation across time) and county-year levels (to allow for data aggregation across sub-counties within year) and are reported in parentheses. Stars indicate statistical significance: * p<.10 ** p<.05 *** p<.01.

Figure C.1: Spatial distributions of population over time



Notes: This figure illustrates the distribution of population (L_{it}) across all locations from 1800 to 2000. The average population in a location in each year is normalized to one. The colors indicate the value, with red indicating a higher population and blue indicating a lower population.

Figure C.2: Spatial distributions of per capita income over time



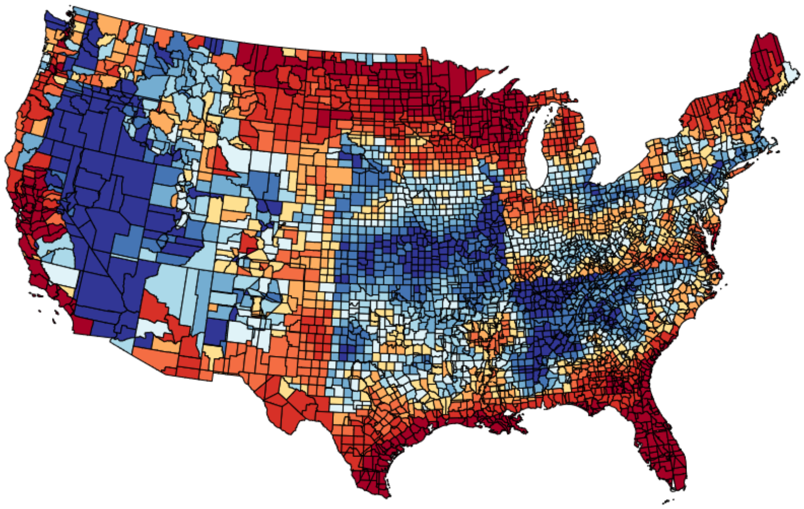
(a) 1850 (b) 1900

(c) 1950 (d) 2000

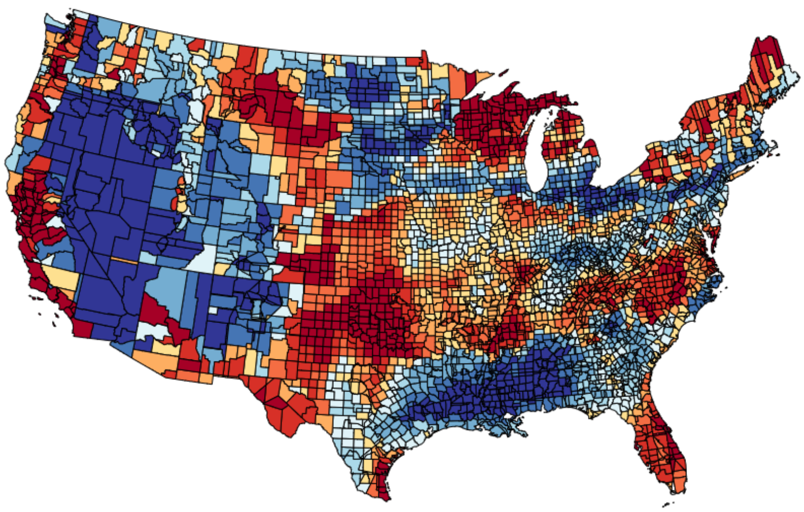
Notes: This figure illustrates the distribution of per capita income (w_{it}) in all locations 1850 to 2000. The average value of w_{it} in a location in each year is normalized to one. The colors indicate the value, with red indicating a higher population and blue indicating a lower wage.

Figure C.3: Estimating productivity and amenity spillovers using plausibly exogenous shifts in labor supply and demand curves over time

(a) Shifts to the supply curve from amenity changes



(b) Shifts to the demand curve from productivity changes



Notes: This figure illustrates the fitted values of the first-stages from the 2SLS regressions in Tables 2 and 3. The left panel shows the predicted change (from 1850-2000) in log population due to plausibly exogenous changes in amenities based on technological improvements which make residing in places with extreme climates of relatively higher amenity value over time. These improvements shift the labor supply curve in each location and can be used to identify the contemporaneous productivity spillover. The right panel shows the predicted trend in log population from plausibly exogenous changes in productivities based on technological improvements and changes in international demand in agricultural production. These improvements shift the labor demand curve in each location and can be used to identify the contemporaneous amenity spillover. Red indicates relatively large values and blue indicates relatively low values.