Problem Set 3

UCSD
ECON 245
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Note: Due Data is March 5.

Problem 0. Eaton and Kortum and the Fréchet Math. Consider the basic Eaton and Kortum model seen in class where the price for a variety ω from origin i in destination j is given by:

$$p_{ij}(\omega) = \frac{w_i}{z_i(\omega)}\tau_{ij}.$$

In each country draws the productivity for each variety $\omega \in [0, 1]$ from a country-specific Fréchet distribution:

$$F_i(z) = \exp\left(-T_i z^{-\theta}\right) \quad T_i > 0 \quad \theta > 1$$

Derive the expression for the fraction of goods that country j buys from country i, i.e., the fraction of goods for which $p_{ij}(\omega) = \min_k\{p_{kj}\}$. Denote this fraction by π_{ij}.
Bonus: Derive the distribution of prices among the goods j imports from j, that is compute the CDF of prices for goods j buys from i. Hint: this should come out to be independent of the origin country i.

Solution P0. First, let's construct the distribution of prices for goods from i in destination j from the distribution of productivities. The probability that the price of a given variety from i in j is lower than some value p is given by:

$$\begin{aligned}
Pr[p_{ij} \leq p] &= Pr[\frac{w_i}{z_i(\omega)}\tau_{ij} \leq p] \\
&= Pr[\frac{w_i}{p}\tau_{ij} \leq z_i(\omega)] \\
&= 1 - F_i(\frac{w_i}{p}\tau_{ij}) \\
&= 1 - \exp\left(-T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta}\right) \\
&\equiv G_{ij}(p)
\end{aligned}$$

The consumer in destination j chooses to a given good from the origin i that offers the lowest price. The probability that consumers in j buy a given good from i is hence the probability that p_{ij} is lower than all other prices $p_{kj} \forall k \neq i$. We now derive an expression for his probability:

$$\begin{aligned}
Pr[p_{ij} \leq \min_{k \neq i}[p_{kj}]] &= \int_0^\infty Pr[\min_{k \neq i}[p_{kj}] > p] Pr[p_{ij} \leq p] dG_{ij}(p) \\
&= \int_0^\infty \prod_{k \neq i}(1 - G_{kj}(p)) dG_{ij}(p) \\
&= \int_0^\infty \prod_{k \neq i}(1 - G_{kj}(p)) g_{ij}(p) dp
\end{aligned}$$

Here $g_{ij}(p)$ is the probability density function, whereas $G_{ij}(p)$ is the cumulative distribution function. The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating using the Fundamental Theorem of Calculus; i.e. given $G_{ij}(p)$:

$$g_{ij}(p) = \frac{dG_{ij}(p)}{dp} = \theta T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta-1}\exp\left(-T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta}\right)$$

Plugging this into the condition above and also using that $\exp(x)\exp(y) = \exp(x+y)$:

$$\begin{aligned}
Pr[p_{ij} \leq \min_{k\neq i}[p_{kj}]] &= \int_0^\infty \exp\left(-p^\theta \sum_{k\neq i} T_k(w_k\tau_{kj})^{-\theta}\right)\theta T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta-1}\exp\left(-T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta}\right)dp \\
&= \int_0^\infty \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta-1}dp \\
&= \int_0^\infty \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta-1}\frac{\sum_k T_k(w_k\tau_{kj})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}}dp \\
&= \frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}}\int_0^\infty \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta(p)^{\theta-1}\sum_k T_k(w_k\tau_{kj})^{-\theta}dp \\
&= \frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}}\left[-\exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\right]_0^\infty \\
&= \frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}}[-0+1] \\
&= \frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}} \\
&\equiv \pi_{ij}
\end{aligned}$$

Now we can look at the distribution of prices among the goods that j ends up buying from i:

$$\begin{aligned}
Pr[p_{ij} \leq p \mid \min_{k}[p_{kj}] = i] &= \int_0^p \frac{Pr[\min_{k\neq i}[p_{kj}] > p]}{Pr[p_{ij} \leq \min_{k\neq i}[p_{kj}]]}dG_{ij}(p) \\
&= \frac{1}{\pi_{ij}}\int_0^p Pr[\min_{k\neq i}[p_{kj}] > p]dG_{ij}(p) \\
&= \frac{1}{\pi_{ij}}\int_0^p \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta T_i(w_i\tau_{ij})^{-\theta}(p)^{\theta-1}dp \\
&= \frac{1}{\pi_{ij}}\frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_k T_k(w_k\tau_{kj})^{-\theta}}\int_0^p \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta(p)^{\theta-1}\sum_k T_k(w_k\tau_{kj})^{-\theta}dp \\
&= 1\int_0^p \exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\theta(p)^{\theta-1}\sum_k T_k(w_k\tau_{kj})^{-\theta}dp \\
&= \left[-\exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)\right]_0^p \\
&= -\exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right)+1 \\
&= 1-\exp\left(-p^\theta \sum_{k} T_k(w_k\tau_{kj})^{-\theta}\right) \\
&\equiv G_j(p)
\end{aligned}$$

Notice how $G_j(p)$ is independent of i! You could have done this derivation for any origin i and would have gotten the same $G_j(p)$. As a result we have shown that the distribution of prices among the products actually bought from an origin is the same across origins for a given destination! Destination countries keep buying from any origin until the average price among goods sourced is the same across origins, for more productive origins this means buying a larger mass of varieties. Lastly, we can exploit the fact that $G_j(p)$ is again a Frechet distribution. Any math book (or Wikipedia) will give you an expression for the mean of a Frechet distribution. The price paid by destination j for the average good its sources is simply the mean of $G_j(p)$ which is:

$$\bar{p}_j = \Gamma(1+\frac{1}{\theta})(\sum_k T_k(w_k\tau_{kj})^{-\theta})^{-\frac{1}{\theta}} \equiv \Gamma(1+\frac{1}{\theta})\Phi_j^{-\frac{1}{\theta}}$$

where Γ is the Gamma function (`https://en.wikipedia.org/wiki/Gamma_function`). Note this is **not** the price index. The price index is not just the mean price, but a geometric mean involving the elasticity of substitution σ. Also note that in many of our other applications where we use Frechet shocks for location choices the average price would be the expected utility before making a moving decision, which would only depend on the origin, not on any particular destination. It would also be the average utility of workers from that origin after choosing locations optimally, by the law of large numbers (since we have a continuum of agents). In our sector choice examples, the average price would be the average wage for a given skill group before choosing sectors but also the average wage of that skill group after choosing a sector, by the law of large numbers (since we have a continuum of agents).

We can also derive the average productivity draws of the goods j buys from i. First compute the CDF of the productivity among goods j buys from i:

$$\begin{aligned}
Pr(z_i < \bar{z} \mid i = \min_k[p_{kj}]) &= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} Pr(z=\bar{z}) \prod_k Pr(p_{ij}(z) < p_{kj} dz \\
&= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta}) \prod_{k\neq i} Pr(\frac{w_i}{z_i}\tau_{ij} < \frac{w_k}{z_k}\tau_{kj}) dz \\
&= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta}) \prod_{k\neq i} Pr(\frac{w_i}{z}\tau_{ij} < \frac{w_k}{z_k}\tau_{kj}) dz \\
&= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta}) \prod_{k\neq i} Pr(z_k < z\frac{w_k}{w_i}\frac{\tau_{kj}}{\tau_{ij}}) dz \\
&= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta}) \exp(-\sum_{k\neq i} T_k(z\frac{w_k}{w_i}\frac{\tau_{kj}}{\tau_{ij}})^{-\theta}) dz \\
&= \frac{1}{\pi_{ij}} \int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta}) \exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k\neq i} T_k(w_k\tau_{kj})^{-\theta}) dz
\end{aligned}$$

Here you need to realize that: $\exp(-T_i z^{-\theta}) = \exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}T_i(w_i\tau_{ij})^{-\theta})$, so that

$$
\begin{aligned}
Pr(z_i < \bar{z} \mid i = \min_k[p_{kj}]) &= \frac{1}{\pi_{ij}}\int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-T_i z^{-\theta})\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k\neq i}T_k(w_k\tau_{kj})^{-\theta})dz \\
&= \frac{1}{\pi_{ij}}\int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})dz \\
&= \frac{1}{\pi_{ij}}\int_0^{\bar{z}} -T_i\theta z^{\theta-1}\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})\frac{(w_i\tau_{ij})^{\theta}\sum_k T_k(w_k\tau_{kj})^{-\theta}}{(w_i\tau_{ij})^{\theta}\sum_k T_k(w_k\tau_{kj})^{-\theta}}dz \\
&= \frac{1}{\pi_{ij}}\int_0^{\bar{z}} -\pi_{ij}\theta z^{\theta-1}\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})(w_i\tau_{ij})^{\theta}\sum_k T_k(w_k\tau_{kj})^{-\theta}dz \\
&= \int_0^{\bar{z}} -\theta z^{\theta-1}(w_i\tau_{ij})^{\theta}\sum_k T_k(w_k\tau_{kj})^{-\theta}\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})dz \\
&= \left[-\exp(-z^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})\right]_0^{\bar{z}} \\
&= 1-\exp(-\bar{z}^{-\theta}(w_i\tau_{ij})^{\theta}\sum_{k}T_k(w_k\tau_{kj})^{-\theta})
\end{aligned}
$$

It is distribution of productivties among the firms from i from which j is buying. We can now use the formula for the mean of the Frechet distribution (see `https://en.wikipedia.org/wiki/Gamma_function`) to simply compute:

$$\bar{z}_{ij} = \Gamma(1+\frac{1}{\theta})((w_i\tau_{ij})^{\theta}\sum_k T_k(w_k\tau_{kj})^{-\theta})^{\frac{1}{\theta}} = \Gamma(1+\frac{1}{\theta})T_i^{\frac{1}{\theta}}\pi_{ij}^{-\frac{1}{\theta}}$$

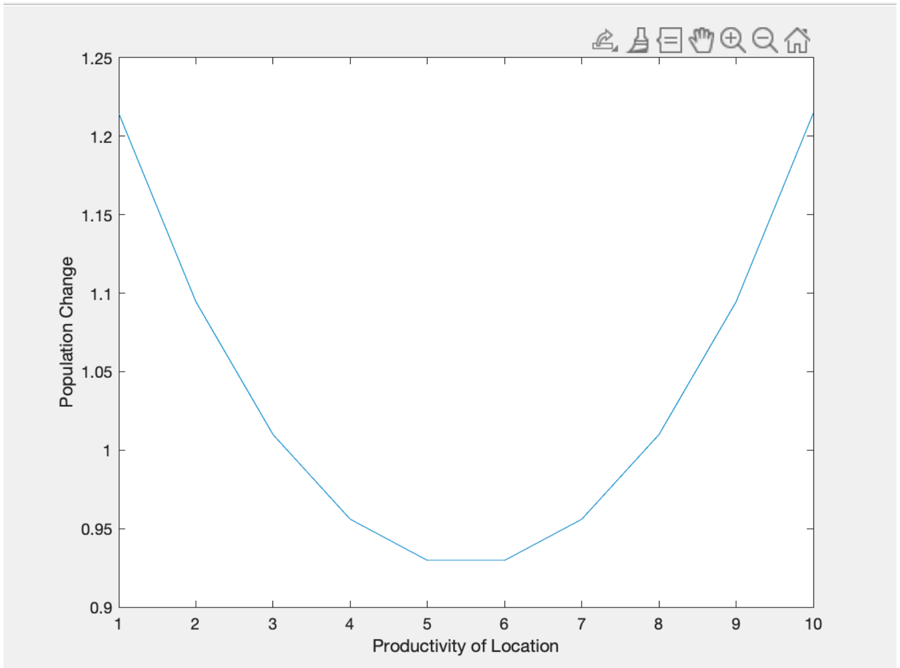
Take the log of this to obtain a linear relationship:

$$\log(\bar{z}_{ij}) = \log(\Gamma(1+\frac{1}{\theta})) + \frac{1}{\theta}\log(T_i) - \frac{1}{\theta}\log(\pi_{ij})$$

First of all: higher average productivity of firms in i, T_i, raises the average productivity of firms that j buys from. **Controlling** for average productivity, we see that the more the two countries trade the lower the productivity of the average firm. The reason is that if, controlling for average productivity j buys a lot from i it **must** be either due to i having low wages or low trade costs with j (the other two components of the price!). In other words if a place has a low wages or low trade costs, j is willing to buy from more low productivity firms, since the productivity disadvantage is made up for by lower wages or trade costs. This is the flip side of the distribution of prices being the same across origins: this directly implies that in locations with low productivity wages or trade cost must be low, to make sure the marginal price is the same across locations.
Bryan and Morten (JPE 2018) use this last formula to compute the average number of efficiency units provided by a worker who decides to move from i to j, i.e., to model selection. The more workers move the higher π_{ij} the lower their average productivity.

Problem 1. The Basic Armington Model with Free Labor Mobility. Consider the exact same Armington model as in the last problem set. Now we introduce one difference: workers can move freely. This introduces an additional equilibrium condition: the real wage $W_i = U_i w_i / P_i$ has to be equal across locations in equilibrium. Solve for the equilibrium with $\tau_{ij} = 2$ and for the one with $\tau_{ij} = 1$ separately. Graph the change in local population between the two scenarios against location productivity. The local amenities, U_i are mirror images of the productivity parameter A_i. The region with $A_i = 1$ has $U_i = 10$, the one with $A_i = 2$ has $U_i = 9$ and so on. Hint: relative to the old code you should now add an "outer loop." First solve for the wage on the inner loop holding population constant. Then compute the real wage in each location.

Figure 1: Trade Cost Reduction with Free Mobility



In the outer loop, you then add some population to locations which have a real wage higher than the median region, and take workers away from regions with a real wage below the median. In this way you iterate between updating wages holding populations fixed, and updating populations holding wages fixed.

Solution P1. Figure 1 shows local employment in each location when trade costs are high, relative to when trade costs are low graphed against a region's productivity. The regions gaining population when trade costs are reduced are the least and most productive.

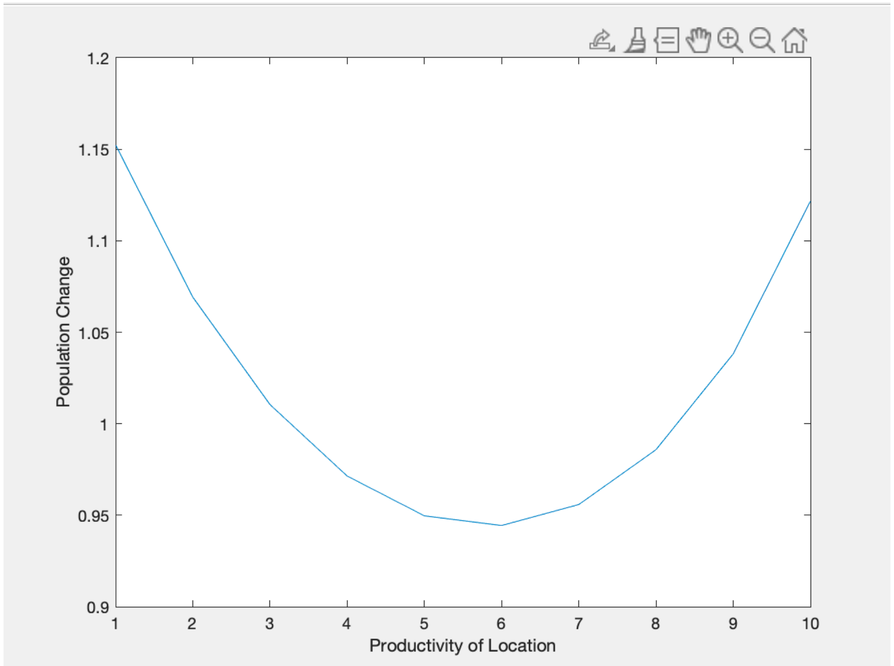
The least productive regions have the highest amenity value. However, in the high trade cost equilibrium their cost of living is high, since local production is not very productive, and so this region buys a large share from other regions which incurs trade cost and raises the price index (i.e., the cost of living). When trade costs are lowered, this region sees a strong reduction in its cost of living since now the price index is the same in all regions. This makes the high amenity value attractive to workers, who migrate in.

The most productive region is initially not very popular since its amenity value is so low. As a result, very few workers want to live there in spatial equilibrium. This drives up the local wage and hence the price for the local good. As a result the local price index is high despite the high productivity of the location! As trade costs fall, the demand for the good produced in this region rises due to its underlying productivity and so some workers move in attracted by high wages.

Overall, with trade costs the middle regions are the most popular, they offer a good combination of decent amenity value and not too high cost of living. These regions loose as trade becomes free: in a world without trade costs "extreme" local fundamentals (amenity, productivity) are in higher demand. The effect of trade liberalization on the spatial distribution of population depends crucially on the correlation between amenity and productivity values.

Problem 2. The Basic Armington Model with Frictional Labor Mobility. Same problem as Problem 1 above but now workers obtain an idiosyncratic preference shock ξ_i for each location which is drawn from a Frechet distribution, $F(\xi) = \exp(-\xi^{-\theta})$. Choose $\theta = 8$. Solve for the equilibrium with $\tau_{ij} = 2$ and for the one with $\tau_{ij} = 1$ separately. Graph the change in local population between the two scenarios against location productivity. Hint: relative to the old code you should now add an "outer loop." First solve for the wage on the inner loop holding population constant. Then compute the real wage in each location.

Figure 2: Trade Cost Reduction with Idiosyncratic Preference Shocks



The outer loop is now simpler: you just need to compute the population distribution implied by the wages. The Frechet assumption gives you an analytical expression for the fraction of workers in each location as a function of the spatial distribution of wages.

Solution P2. The economic intuition with the Frechet shock is the same as before. However, a Figure 2 shows the population changes are less pronounced. The reason is that workers are less responsive to the real wage (including amenities, wages, cost of living) since they have their own reasons for their location choices. Some workers relocate when wages rise in the most productive region, but fewer than above because many prefer to live in other regions due to unobserved proprietary reasons.